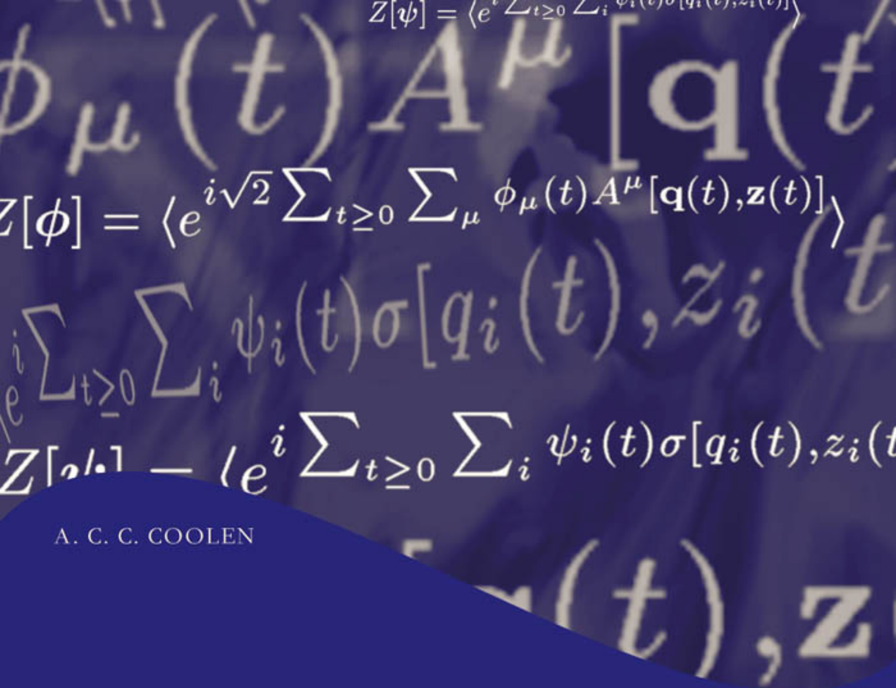

A. C. C. COOLEN

The Mathematical Theory of Minority Games

statistical mechanics of interacting agents

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Statistical mechanics of interacting agents

A.C.C. COOLEN

King's College London

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To Karin, Abel and Finn

Preface

This book describes Minority Games and explains their mathematical analysis from the viewpoint of a discipline-hopping non-equilibrium statistical mechanicist. There are several reasons why a textbook on Minority Games might be helpful at this point in time. Firstly, it appears that popularity and scope of Minority Games are still growing. A textbook might assist those researchers and students in theoretical physics, applied mathematics or economics who wish to enter this scene by offering an accessible but detailed and explicit introduction to their mathematical analysis, uniform in style and mathematical notation, selective in its choice of topics, and up to date. It might also serve those who are already on the scene by showing how different mathematical approaches are related, and by acting as an efficient point of departure for new research.

The author has tried to exploit the specific advantages offered by writing a textbook as opposed to writing research or review papers. The text aims to be self-contained in explaining detailed mathematical derivations, including any hidden or discipline-specific assumptions and the mathematical methods on which these derivations rely. Rather than introducing and explaining mathematical formalisms such as replica theory and generating functional analysis by first turning to spin models (which indeed generated these tools, but which might put off those who are interested in market models only), in the present text an attempt is made to explain such formalisms 'en passant', with subtleties discussed in footnotes and with references for further reading. Furthermore, various new derivations have been added with the objective to clarify, connect and supplement the existing analyses and theoretical results in literature.

The book is organized with the benefit of hindsight — results are presented in the order expected to be most beneficial to the reader, rather than strictly chronologically — and the author took the textbook style as a justification for being relieved of the obligation to give an encyclopedic account of the vast numbers of papers written on Minority Games, and to concentrate instead on explaining the emerging mathematical theory. The main references are concentrated and discussed in a bibliography section. Also the details of numerical simulation experiments are given in an appendix, in order to prevent unnecessary distraction in figure captions. In its choice of mathematical notation the text tries to stay as close as possible to that of the original research papers, in as much as this does not conflict with the desire for uniformity, while accepting that the limited combined size of the Roman and Greek alphabets makes a certain degree of symbol recycling inevitable.

A.C.C.C.

August 2004 *London*

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Contents

1. Introduction

Minority Games (MGs) are relatively simple mathematical models that were initially designed to increase our understanding of the cooperative phenomena and fluctuations that are observed in financial markets. They apply in fact more generally to any situation where a large number of selfish decision-making 'agents' all try to achieve individual gain by predicting (inductively, on the basis of incomplete public information, and often with an element of irrationality) and anticipating the decisions to be taken by their competitors. Such situations arise quite frequently, e.g. in queuing and traffic problems, in politics, sociology, etc. Not just in markets. Although there is a considerable history of mathematical studies aimed at understanding the interaction between agents in socio-economic systems, such as game theory, the key advantage of models such as MG is that in the latter one does not assume or rely on the participants in these systems to act fully rationally.

The basic MG has come to be seen as the minimal mathematical model that contains the essential ingredients thought to be responsible for many of the non-trivial macroscopic phenomena observed in the above types of competitive socio-economic scenarios. It is not difficult to write pages and devise ad hoc phenomenological descriptions to 'explain' the behaviour of models that from the start are so detailed that they can only be studied numerically. The power and beauty of the MG is that, although it produces intriguing macroscopic phenomena, its microscopic rules are too simple and explicit to justify the easy option. The model simply begs to be analyzed and solved properly. It has thus become the 'harmonic oscillator' of its field, much like the Sherrington–Kirkpatrick model became the 'harmonic oscillator' for the study of disordered physical systems.

For mathematicians and theoretical physicists, MGs are intriguing disordered systems, residing under the fashionable banner of 'econo-physics'. More specifically, finding macroscopic mathematical laws from microscopic equations (usually stochastic) describing a system of many structurally similar interacting elements is the province of statistical mechanics. It is irrelevant for the statistical mechanicist whether the microscopic variables at hand represent coordinates of colliding particles, microscopic magnets, reptating polymers, or (as in the present case) the actions of decision-making agents. In contrast to most physical systems, however, the stochastic equations of MGs do not evolve towards an equilibrium state, so a proper mathematical analysis normally requires solving their dynamics. This task was found to be surprisingly non-trivial.

In this introductory chapter we first try to answer the two most basic questions: what is MG, and why would we want to study it? The answers to these questions depend on one's background. For those aiming to understand socio-economic systems, MG is a mathematical implementation of the so-called El Farol bar problem, which is worthy of study because it exhibits intriguing collective phenomena that would not emerge in game-theoretic market models, yet it is very easy to simulate on a computer. To the theorist, MG is an appealing mathematical challenge: at first sight it appears to be just a simple (if unusual) disordered statistical mechanical system with a non-equilibrium phase transition, but it turned out to be surprisingly non-trivial and succeeded in resisting solution for quite some time.

1.1 Inductive reasoning: the El Farol bar problem

A considerable amount of mathematical literature has been generated over the years under the name of 'game theory'. Here the word game should not be taken literally. It is chosen to describe any form of competitive dynamic interaction between multiple individuals, where each aims for personal gain (defined in some precise mathematical way), but where that gain depends crucially on the actions taken by the other participants. This definition describes not only games in the literal sense, but also many other areas of collective human activity, such as trading, politics, and even the conduct of wars. The mathematically interesting situations are those where it is fundamentally impossible for the individual objectives of all players in the game to be achieved simultaneously. In game theory one is concerned with finding optimal playing strategies, which can be defined either with a view to achieving the maximum average gain over the community of players as a whole or from the more selfish perspective of optimizing an individual's own situation. In either case, statements will have to be made regarding the anticipated actions and responses by the competitors. In traditional game theory this is done by assuming perfect deductive rationality; each player is assumed to have (i) enough information and (ii) the frame of mind to base all his decisions in the game on precise logical reasoning.

Unfortunately, in real world situations neither assumption (i) nor assumption (ii) is usually correct. Especially in markets, which is the original domain to be mimicked by MGs, one is as a rule not able to reason one's way fully logically towards optimal trading actions. There will of course be certain rational guidelines, but most traders have to rely in practice largely on experience and intuition (if not partly guessing), or on copying the actions of colleagues who they suspect of having such qualities. As a result, many quantitative phenomena observed in market-related time series (averages and fluctuations of prices and traded volumes, temporal correlations, etc.), being largely the result of inductive and partly irrational decision making by interacting individuals, have at most been only partly understood.

The El Farol bar problem was invented to illustrate the fundamental differences between logical–deductive and (partially) irrational–inductive reasoning by groups of interacting individuals. The set-up of this problem is as follows. The El Farol bar in Santa-Fé (which exists, but is in fact a restaurant) organizes Irish music on Thursday nights, which is a very popular weekly event but at times leads to overcrowding. Let us image that a number of residents of Santa-Fé decide independently each Thursday whether or not to go to El Farol. If, however, the number of visitors exceeds a certain number (say 50), the place will be overcrowded and the evening will be unpleasant. Thus, in order to decide whether or not to go, each individual will effectively have to guess the decisions to be taken by others, and then do the opposite of the majority (i.e. go when most plan to stay at home, and vice versa). Here deductive reasoning is clearly ruled out. Even if it were possible to predict the thinking of others, there cannot exist a correct logical rule with which to work out how many would go to El Farol on a given day. Such a rule would ultimately be picked up and put to use, and thereby invalidate itself. Instead, each individual will in practice probably rely on one of a number of ad hoc and personal rule-of-thumb methods (or ‘predictors’) to guess whether or not El Farol will be crowded on a given Thursday evening, e.g. by using extrapolation of the last three weeks’ attendance, or by taking into account whether it is the last week of the month, or perhaps based on the average temperature of the day, etc. Each individual keeps track of the performance of his personal predictors over time, and uses at any given time the predictor that at that point has the best track record.

So, within this scenario, what can one expect to happen? Given the individual predictors of our participants, and given an initial state (here: the choices of the predictors used at the start), the decisions of whether or not to go to El Farol will evolve deterministically over time. However, this does not mean that a trivial attendance pattern will emerge. Patterns, after all, imply potential for attendance prediction; so provided the pool of prospective visitors is large and their predictors are sufficiently diverse, one expects that any simple pattern will eventually be picked up by someone and thereby destroyed. Thus, the average attendance of El Farol should fluctuate around the critical value of 50; if the number of potential visitors is odd there cannot even be a stationary state. Yet, numerical simulations of the El Farol bar problem show that decision making will not be random either. The essence of the problem is that the quality of any given predictor is defined fully in terms of the decisions taken by others. Consequently, if one individual changes his choice of predictor, this will influence the attendance at El Farol and thereby affect the performance of the predictors presently used by all others. This, in turn, can initiate an avalanche of predictor switches in the population. The result of all this is a highly non-trivial collective dynamical process.

The El Farol bar problem is a nice and simple illustration of the dynamical complexity of socio-economic systems defined by large numbers of interacting decision-making individuals, such that each aims for personal gain but with that gain being defined in terms of the decisions to be taken by others, and where logical–deductive decision making is not an option. Here all players try to take decisions that differ from those taken by the majority (i.e. stay at home when most others will go to El Farol, and go when most others stay at home). Such situations are far from unusual. In markets one might imagine speculating traders who have to decide whether to buy or sell an asset, and where the resulting price will be high if the sellers are in the minority but low if the sellers are in the majority. Similarly, in traffic one would often wish to select from a number of options a route where the least amount of traffic is expected (but so do the other motorists $\cdots$). Similar problems can be expected to emerge in Internet traffic.

MGs, in a nutshell, are clever mathematical implementations of the El Farol bar problem, which have been designed with the objective to enable application of the mathematical techniques of statistical mechanics. In this latter field there is about a century of experience in extracting exact mathematical equations and predictions on the collective behaviour of large systems that are composed of structurally similar interacting microscopic elements (usually molecules, atomic magnets, polymers, etc.), from the underlying microscopic physical laws. Much of statistical mechanics is concerned with the detection and analysis of phase transitions. These are drastic changes in macroscopic behaviour that are observed at specific critical values of the model's control parameters, and that are clearly reminiscent of the critical phenomena observed in markets (e.g. crashes) and in traffic (e.g. jams). Although physics has been their traditional domain, the *mathematical methods* of statistical mechanics are in no way restricted in their application to physical systems. There is no reason why they should not apply equally well to socio-economic ones.

1.2 Standard MG

Let us first define the traditional and basic MG as it was first introduced. Extensions and generalizations will come later. This model is a direct mathematical implementation of the key elements of the El Farol bar problem. However, in line with established convention we will now switch to 'market' language: the bar visitors become 'agents', the binary decisions (going to El Farol versus staying home) become 'buy' or 'sell', the predictors become 'strategies', the possible experiences pleasant versus overcrowded become 'profit' or 'loss', and the total bar attendance becomes the 'total market bid'. We will also choose our notation to be as close as possible to that adopted in literature. As in the El Farol bar problem, the agents in the MG aim to take those decisions that will put them in the minority group among their peers, since the market favours the sellers when most agents choose to buy, but it favours the buyers when most choose to sell. Hence the name MG.

Let there to be N agents in the game, labelled by a Roman index $i = 1, \ldots, N$. This number N, usually taken to be odd, is assumed to be large. The game proceeds at discrete iteration steps, which we denote by $\ell = 0, 1, 2, 3, \ldots$. We now translate the main ingredients of the El Farol bar problem into simple mathematical rules. These ingredients are:

(i) *Agents' actions*
At each step ℓ each agent i takes a binary decision $b_i(\ell) \in \{-1, 1\}$. This is called a 'bid', representing e.g. buying (-1) versus selling $(+1)$. A re-scaled (aggregate) overall bid at step ℓ is defined as $A(\ell) = N^{-1/2} \sum_{i=1}^{N} b_i(\ell)$.

(ii) *Public information*
Agents base their decisions on historical market information. They are given at step ℓ the *signs* of the overall bids (i.e. whether buyers or sellers were the minority group) over the M most recent steps, i.e. the string

$$(\text{sgn}[A(\ell - M)], \ldots, \text{sgn}[A(\ell - 1)]) \in \{-1, 1\}^M.$$

(iii) *Agents' strategies*
Each agent i has S strategies $\mathbf{R}^{ia}$, with $a = 1, \ldots, S$. A strategy $\mathbf{R}^{ia}$ defines a mapping from information strings to a recommended trading action:

$$\mathbf{R}^{ia} : \{-1, 1\}^M \to \{-1, 1\}.$$

A strategy functions as a look-up table with 2^M entries, each being ± 1. These entries remain fixed throughout the game. If agent i uses strategy a at step ℓ, then he will act (deterministically) according to

$$b_i(\ell) = R^{ia}(\text{sgn}[A(\ell - M)], \ldots, \text{sgn}[A(\ell - 1)]).$$

To obtain closed equations, we must finally specify how our agents determine which of their S personal strategies to use. A good strategy is one that would have prescribed many minority decisions. Given the above definitions, a strategy $\mathbf{R}^{ia}$ is good at stage ℓ if $R^{ia}(\text{sgn}[A(\ell - M)], \ldots, \text{sgn}[A(\ell - 1)]) = -\text{sgn}[A(\ell)]$. Hence

(iv) *Strategy valuations*
Each agent i keeps track of a valuation p_{ia} for each of his strategies, measuring their track records (irrespective of whether they were actually used):

$$p_{ia}(\ell+1) = p_{ia}(\ell) - A(\ell)\, R^{ia}(\text{sgn}[A(\ell-M)], \ldots, \text{sgn}[A(\ell-1)]).$$

(v) *Dynamic strategy selection*
At each step ℓ of the game, each agent i will select his best strategy $a_i(\ell)$ at that stage of the process, defined as $a_i(\ell) = \arg\max_{a\in\{1,\ldots,S\}}\{p_{ia}(\ell)\}$.

The logic of this construction is illustrated in Fig. 1.1, for $S = 3$. The entries of the look-up tables $\mathbf{R}^{ia}$ are all drawn independently at random from $\{-1, 1\}$ (with equal probabilities) before the start of the game. They constitute frozen disorder in the problem.

The basic microscopic degrees of freedom in the MG are the strategy valuations $\{p_{ia}\}$. Upon combining the above model ingredients (*i*)–(*v*), these valuations are seen to evolve according to the following coupled and non-linear (but deterministic) equations:

$$p_{ia}(\ell+1) = p_{ia}(\ell) - A(\ell)\, R^{ia}(\text{sgn}[A(\ell-M)], \ldots, \text{sgn}[A(\ell-1)]), \quad (1.1)$$

$$A(\ell) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} R^{ia_i(\ell)}(\text{sgn}[A(\ell-M)], \ldots, \text{sgn}[A(\ell-1)]), \quad (1.2)$$

$$a_i(\ell) = \arg\max_{a\in\{1,\ldots,S\}}\{p_{ia}(\ell)\}. \quad (1.3)$$

Two minor tie-breaking issues have been glanced over so far. Firstly, when the number N is odd the function sgn$[A(\ell)]$ is well-defined, but for even N we must agree on how to assign a value from the set $\{-1, 1\}$ to sgn$[0]$. This is done randomly, with equal probabilities for ± 1. Similarly, if at stage ℓ of the game and for some agent i there exist two (or more) different strategies $a, a' \in \{1, \ldots, S\}$ with $p_{ia}(\ell) = p_{ia'}(\ell)$, we will select $a_i(\ell)$ (the strategy to be used by agent i at step ℓ) randomly from a and a' with equal probabilities.

1.3 Phenomenology of MG

The MG achieved its popularity in the relevant scientific communities because it exhibits remarkable collective behaviour, in spite of the apparent simplicity of its definition. In this section, we describe the phenomenology of the MG, which has been observed in numerical simulations of the process (1.1)–(1.3). In selecting experimental conditions and quantities to be measured we make use of hindsight. In addition, this

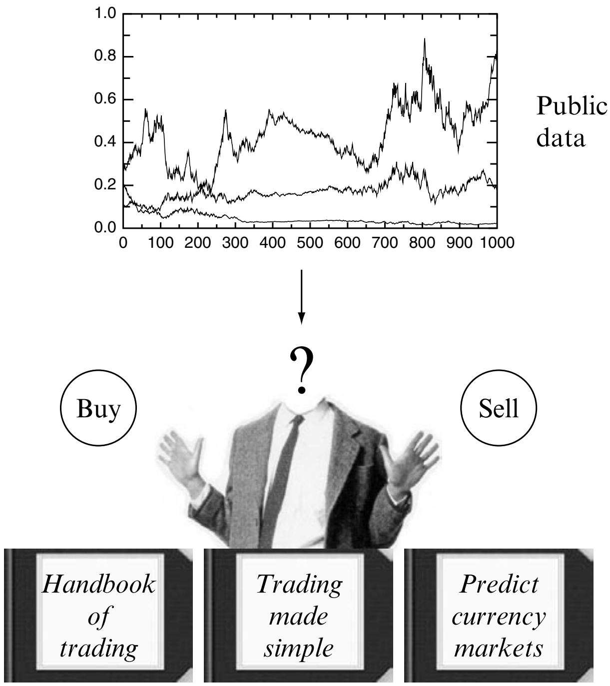

Fig. 1.1 The logical structure of MG. At each point in time each agent observes public data on the recent behaviour of the market, and is asked to convert this observation into one of two possible actions: buy or sell. To do so, every agent has S personal look-up tables ('strategies'), each prescribing a response to every possible market history (here $S = 3$). Given a strategy, the action of an agent is therefore deterministic. Agents make profit when they take a minority decision, i.e. choosing 'buy' when most decide to sell, or vice versa. The dynamics of the MG refers to the choice of strategies to be used by each agent from their arsenals. This choice is based on monitoring how often each strategy would have led to a profitable (i.e. minority) decision.

numerical exploration will give us valuable intuition and guide us later on how to solve the model using statistical mechanical techniques.

1.3.1 Statistical properties of the overall bid

We first turn to the statistical properties of the re-scaled overall market bid $A(\ell) = N^{-1/2} \sum_i b_i(\ell) \in \mathbb{R}$, as it is observed upon iterating equations (1.1)–(1.3) numerically. A typical simulation result is depicted in Fig. 1.2, where we show the observed values of $A(\ell)$ (top row, values measured at intervals of 80 iterations) as well as the corresponding histograms of these values, plotted logarithmically (bottom row). The initial strategy valuations $p_{ia}(0)$ were all drawn randomly, very close to zero. To aid

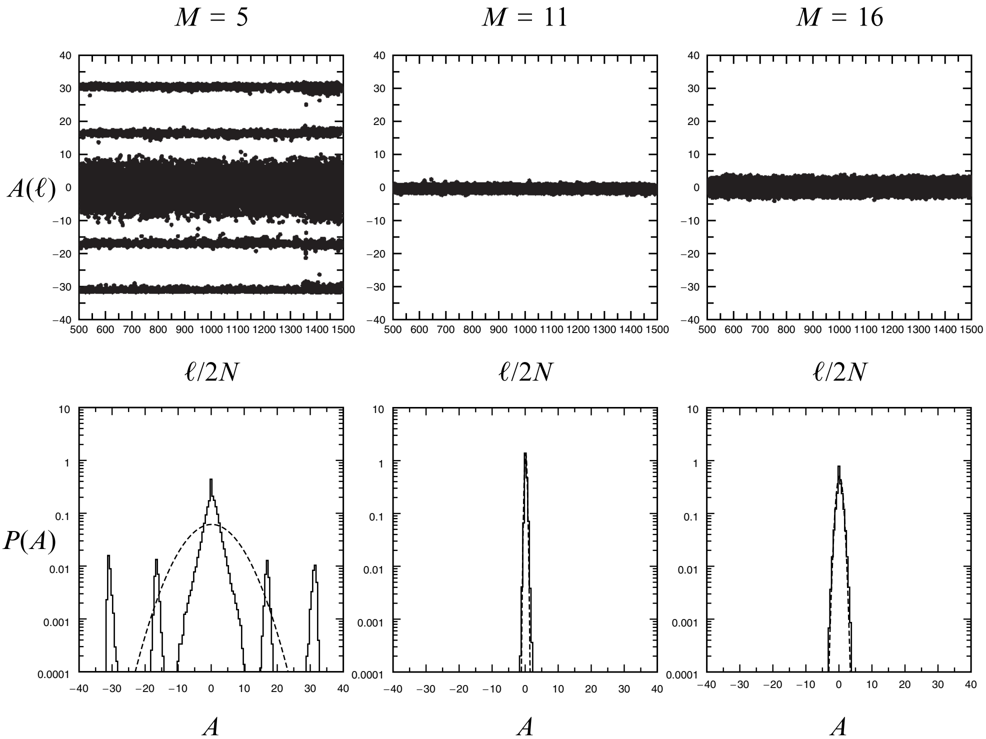

Fig. 1.2 Top row: numerical simulation measurements of the re-scaled total bids $A(\ell)$, for the case where each agent has $S = 2$ strategies. History depths: $M = 5$ (left), $M = 11$ (middle), and $M = 16$ (right). Bottom row: the corresponding distributions $P(A)$ of these observed values (represented as histograms, and plotted logarithmically), together with Gaussian distributions (dashed) of width and average identical to those of the observed $P(A)$. It is immediately clear that at least for small M the individual bids $b_i(\ell)$ contributing to the re-scaled sum $A(\ell)$ must have non-trivial correlations.

interpretation of these data, the histograms are drawn together with the values corresponding to Gaussian distributions with moments (i.e. average and variance) identical to those measured in the data for the overall bids. Such bid data are not found to be very sensitive to a further increase in the system size N beyond the values used in generating Fig. 1.2, viz. $N = 4097$. It is also instructive to inspect the relation (if any) between the overall bids $A(\ell)$ at successive iteration steps of the game. This can be done simply by plotting the pairs $(A(\ell - 1), A(\ell))$ as points in a plane (see e.g. Fig. 1.3). We may summarize what we observe in these simulation experiments as follows:

- The values of the overall bids $A(\ell)$ are distributed around zero, with a variance of order N^0. This confirms *en passant* the wisdom of the specific scaling with the

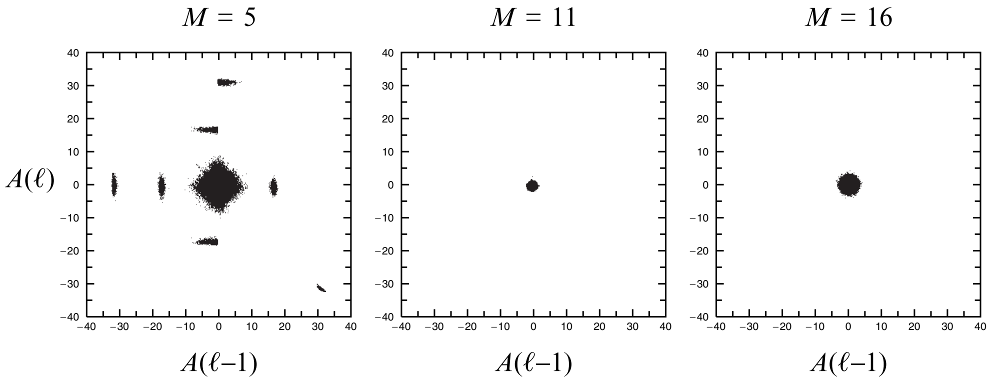

Fig. 1.3 Here we show the relation between the values of the re-scaled total bid at successive iteration steps in the simulations of the previous figure, in plots of $A(\ell)$ versus $A(\ell-1)$. These data again underline the surprisingly non-trivial nature of the dynamics for small M.

system size N in their definition. A non-zero average for the overall bids would have implied immediate potential for prediction, which would have been picked up by the agents and eliminated.

- For small values of the history depth M, the fluctuations in the overall bid $A(\ell)$ around its average value are strongly non-Gaussian. This suggests that the fluctuations in the constituent individual bids $b_i(\ell)$ must be correlated in some complicated way.
- The magnitude of the fluctuations in the overall market bid depends non-monotonically on the history depth M, with large fluctuations for small and for large M, but with smaller fluctuations at intermediate values. This is a remarkable and unexpected observation, for which there is no obvious explanation as yet.
- The low M region with large fluctuations reveals a complicated structure in terms of the relation between the overall bid values $A(\ell)$ at successive iteration steps, which goes beyond simple sign alternation.

In any context, but especially for (financial) markets, large fluctuations of the re-scaled overall bid $A(\ell)$ are highly undesirable in that they imply a waste of resources. Let us therefore probe the MG further by inspecting these fluctuations, with their unexpected non-monotonic behaviour, in more detail.

1.3.2 Volatility and identification of control parameters

In the statistical analysis of financial time series one usually defines the average $\langle A\rangle$ and the volatility σ for a time series $\{A(\ell)\}$, as follows

$$\langle A\rangle = \lim_{L\to\infty}\frac{1}{L}\sum_{\ell=1}^{L} A(\ell), \quad \sigma^2 = \lim_{L\to\infty}\frac{1}{L}\sum_{\ell=1}^{L}[A(\ell)-\langle A\rangle]^2. \tag{1.4}$$

The volatility σ measures the magnitude of the fluctuations in the overall market bid in the stationary state; the definitions (1.4) therefore make sense only if the process approaches stationarity. As a yardstick against which to measure the actual values that we find in the MG, it will be helpful to first consider the case of purely random decision making. If we were to simply draw each individual bid $b_i(\ell)$ independently at random from $\{-1, 1\}$ at each point in time, so that $\langle b_i\rangle = 0$ and $\langle b_i b_j\rangle = \delta_{ij}$, we would find

$$\langle A\rangle = \frac{1}{\sqrt{N}}\sum_i \langle b_i\rangle = 0, \tag{1.5}$$

$$\sigma^2 = \lim_{L\to\infty}\frac{1}{L}\sum_{\ell=1}^{L} A^2(\ell) = \frac{1}{N}\sum_{ij=1}^{N}\langle b_i b_j\rangle = \frac{1}{N}\sum_{i=1}^{N} 1 = 1. \tag{1.6}$$

We conclude that an MG with $\sigma > 1$ may be seen as describing an inefficient market, and one with $\sigma < 1$ an efficient market.

Having established a benchmark, let us next measure the volatility in numerical simulations (we note that Fig. 1.2 suggests that the system does approach a stationary state in terms of the overall bids), again following random initializations of the strategy valuations, close to zero. On the basis of Fig. 1.2 we must expect the volatility to depend non-monotonically on the history parameter M. The outcome of measuring σ in simulations is quite surprising. Furthermore, if the volatility data are plotted cleverly, they give us important clues on how to set up our statistical mechanical theory later. Figure 1.4 shows the volatility when plotted as a function of the ratio $\alpha = 2^M/N$, and for different values of both the number S of strategies per agent and the number N of agents. From this figure we may conclude:

- For small values of the ratio $\alpha = 2^M/N$ one observes a highly inefficient market, with the agents performing much worse than if they had taken purely random decisions. For intermediate values one obtains an efficient market. For large values of $\alpha = 2^M/N$ the agents' performance approaches that corresponding to random decision making.
- The volatility curves obtained for different system sizes N, but with an identical number S of strategies per agent, collapse nearly perfectly when plotted as a function of the ratio $\alpha = 2^M/N$. This tells us that for statistical mechanical calculations, where one analyzes the limit $N \to \infty$, the quantity α will be the relevant control parameter.

- Changing the value of the number of strategies per agent S, provided it remains finite compared with N, does not change the volatility curves qualitatively but seems to induce only a shift and a re-scaling for small α.

One would probably not exaggerate in saying that the volatity curves as shown in Fig. 1.4 were solely responsible for the initial popularity of the MG, albeit for different reasons in the relevant communities. To those interested in understanding real (financial) markets they seemed to indicate that, at least for intermediate values of the ratio α, the agents in the MG develop expertise: they can and do learn (implicitly) to *predict* the decisions to be taken by their competitors and hence predict the market, solely on the basis of experience. To statistical mechanicists, on the other hand, Fig. 1.4 suggested the existence of a non-equilibrium phase transition at some critical value α_c. Initially, nobody could explain the non-monotonic dependence of the volatility in the MG on the parameter α, neither in terms of market phenomenology nor on mathematical grounds. This mystery added significantly to the MG's appeal. Since the behaviour of the MG is seen not to depend qualitatively on the number S, we feel justified in limiting ourselves whenever convenient to the simplest non-trivial case $S = 2$, where each agent has just two personal strategies to select from.

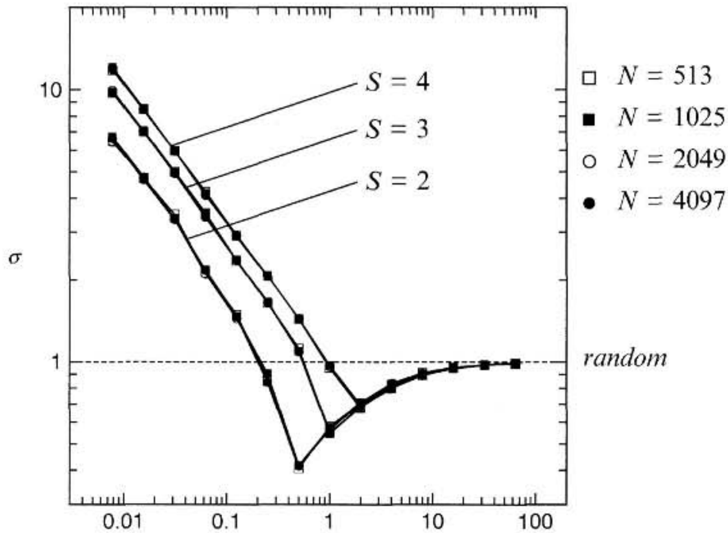

Fig. 1.4 Simulation measurements of the volatility $\sigma = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$ of the re-scaled total bid, shown in log–log plots as functions of the parameter $\alpha = 2^M/N$ for different system sizes N and different numbers S of strategies per agent. The line segments connecting the markers are guides to the eye; only the markers correspond to actual data. Dashed line: the benchmark value $\sigma = 1$ which would have been found for random decision making.

1.3.3 The concept of 'frozen' agents

In an attempt to unravel the mystery of the non-monotonic behaviour of the volatility by further numerical experimention, one could probe more subtle dynamical properties of the MG. Since the most important microscopic state changes in the MG are those where agents switch strategy, one could measure the frequency of such strategy switches in the population of agents, in the stationary state. Let us denote by $f_i \in [0,1]$ the fraction of iterations in the stationary state where agent i changes his active strategy, and by $W[f]$ the distribution of these N values $\{f_i\}$ over the community of agents, i.e.

$$f_i = \lim_{L\to\infty} \frac{1}{L} \sum_{\ell=1}^{L} [1 - \delta_{a_i(\ell),a_i(\ell-1)}], \quad W(f) = \frac{1}{N} \sum_{i=1}^{N} \delta[f - f_i], \tag{1.7}$$

where $\delta[x]$ denotes the δ-function, see Appendix A. In simulations one typically finds distributions of the form shown in Fig. 1.5, which seem to suggest that for large and small values of the market history M all agents do change strategy now and then (albeit at different frequencies), but that for intermediate values of M there is a considerable fraction of agents that in the stationary state do not change strategy at all. Such agents will be called 'frozen', and the fraction of such agents is written as ϕ, i.e.

$$\phi = \frac{1}{N} \sum_{i=1}^{N} \lim_{L\to\infty} \delta_{L,\sum_{\ell=1}^{L} \delta_{a_i(\ell),a_i(\ell-1)}}. \tag{1.8}$$

If we plot the value ϕ of this fraction, as observed in simulations, as a function of the control parameter $\alpha = 2^M/N$, we find an intriguing result, which strengthens further the case for there being a non-equilibrium phase transition for some critical value α_c. We see that

- There will generally be a non-zero fraction ϕ of agents in the stationary state who will *never* change their active strategy ('frozen agents').
- For small and large values of the control parameter $\alpha = 2^M/N$ this fraction ϕ vanishes, but at intermediate values of α it becomes large.
- The fraction ϕ of frozen agents appears to exhibit a discontinuity at the same value of α where the volatility has its minimum, jumping from zero (for small α) to its maximum value.

Again, there appears to be no obvious explanation yet for this particular behaviour, but for the statistical mechanicist the cumulative evidence for a phase transition at some α_c has by now become quite conclusive.

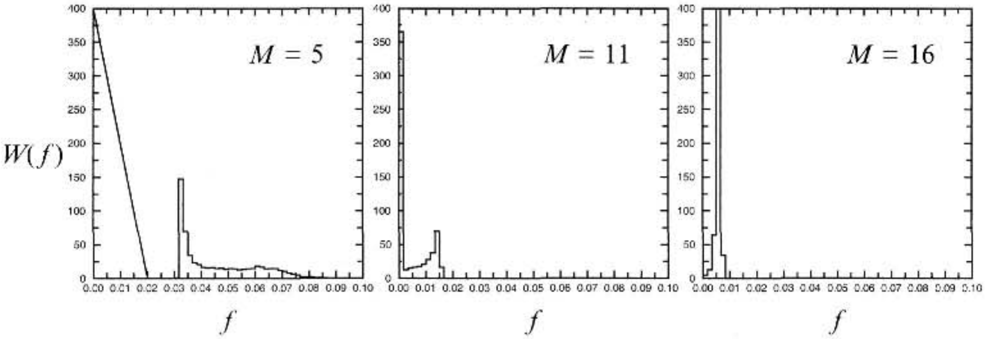

Fig. 1.5 Numerical simulation measurements of the distributions $W(f) = N^{-1} \sum_{i \leq N} \delta[f - f_i]$ (shown as histograms) of the fractions $f_i \in [0, 1]$ of MG iterations for which individual agents i are found to change their active strategy, in the stationary state and with $S = 2$ strategies per agent. History depths: $M = 5$ (left), $M = 11$ (middle), and $M = 16$ (right).

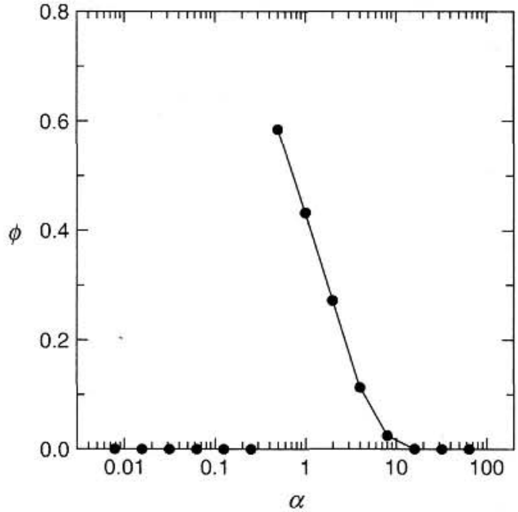

Fig. 1.6 Simulation measurements of the fraction ϕ of 'frozen' agents, i.e. those who never change strategy in the stationary state (and hence have $f_i = 0$ in the experiments of Fig. 1.5).

1.3.4 Non-ergodicity of MG

The relevance of one particular aspect of the simulation experiments underlying the simulation data shown in our figures appeared to have been overlooked initially in MG literature. These experiments were all performed following the initialization $p_{ia}(0) \approx 0$ for all i and all S. This 'tabula rasa' initialization implies that before observing

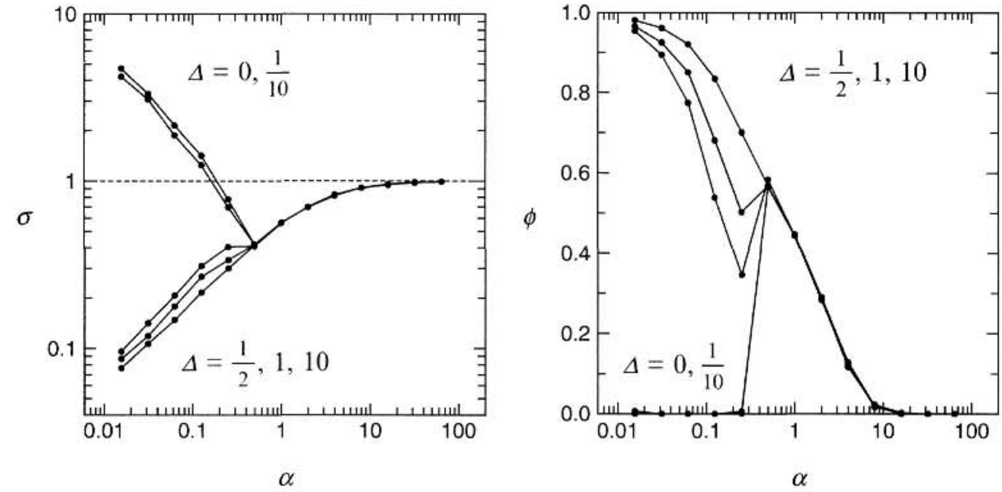

Fig. 1.7 Simulation measurements of the volatility $\sigma = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ of the re-scaled total bid (left) and of the fraction ϕ of 'frozen' agents (those who never change strategy, i.e. who have $f_i = 0$ in the notation of Fig. 1.5, right) as functions of the parameter $\alpha = 2^M/N$. Each agent has $S = 2$ strategies. Initialization: $|p_{i1}(0) - p_{i2}(0)| = 2\Delta$ for all i, with $\Delta \in \{0,\ 0.1,\ 0.5,\ 1,\ 10\}$ (from top to bottom in the left panel, and from bottom to top in the right panel). The line segments connecting the markers are guides to the eye. Horizontal dashed line (left): the benchmark value $\sigma = 1$ which would correspond to random decision making. For small α one observes a strong dependence of both the volatility and the 'frozen' fraction on initial conditions.

any market data all traders value each of their strategies equally. This choice seemed sensible at the time, but was found to have important implications.

Let us see what happens if for $S = 2$ instead we choose each initial valuation $p_{ia}(0)$, which represents the a priori confidence of agent i in his ath personal strategy, such that $|p_{i1}(0) - p_{i2}(0)| = 2\Delta > 0$ for each i. Now the agents are already biased towards one randomly drawn strategy at the start, to an extent that is controlled by the parameter Δ. For $\Delta \to 0$ we return to the previous 'tabula rasa' initial conditions. For nonzero values of Δ we find yet further surprising results if we measure, for instance, the volatility σ, the fraction of frozen agents ϕ, or the distribution $P(A)$ of overall market bids, see Figs 1.7 and 1.8:

- Above the value for α, which we have come to believe marks a critical point, the volatility and the fraction of frozen agents are not sensitive to initial conditions. There the system appears to be ergodic, i.e. not confined to a restricted region of configuration space simply by initial conditions.
- Below this value for α, in contrast, both the volatility and the fraction of frozen agents are strongly dependent on initial conditions. The volatility σ decreases with

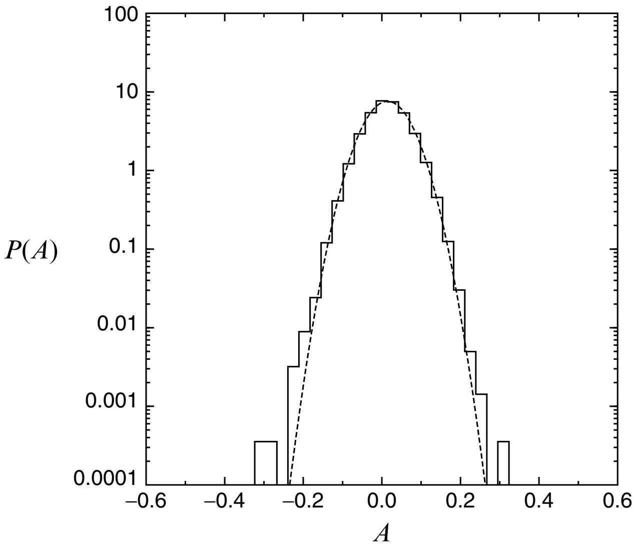

Fig. 1.8 Statistics of the overall bids $A(\ell)$ as measured in numerical simulations, for $S = 2$ (two strategies per agent) and $M = 5$, but now recorded in the low volatility state (following highly biased initial strategy valuations, with $\Delta = 10$). The simulation data are plotted as histograms, and are shown together with Gaussian distributions (dashed) of width and average identical to those of the observed $P(A)$ (note the different scale of the horizontal axis, compared with those in Fig. 1.2, where we had $\Delta = 0$).

increasing Δ (i.e. biased initial strategy valuations lead to a lower volatility), but the frozen fraction ϕ increases with increasing Δ (i.e. biased initial strategy valuations lead to an increase in the fraction of frozen agents).

- The different low α curves for the volatility can be described nearly perfectly by power laws, with $\sigma \sim \alpha^{-1/2}$ for small Δ (the high volatility branches) and $\sigma \sim \alpha^{1/2}$ for large Δ (the low volatility branches).
- In the low volatility solutions (i.e. those that follow biased initial strategy valuations, at small values of α) the overall bid distribution $P(A)$ is Gaussian, in contrast to the bid distributions that are observed in the high volatility solutions (which were shown e.g. in Fig. 1.2).

The observed dependences of σ, ϕ, and the bid distribution $P(A)$ on initial conditions for small α are not just simple transient effects. Of this one can easily convince oneself by further experimentation with different durations of simulation experiments and different system sizes. It can also be inferred from the simple fact that at time zero there cannot yet be any correlations between the strategy valuations and the look-up table entries $\{R_\mu^{ia}\}$, so that at the start of the game one must always find the random trading expectation values $\langle A\rangle = 0$ and $\langle A^2\rangle = 1$, for any value of the bias Δ.

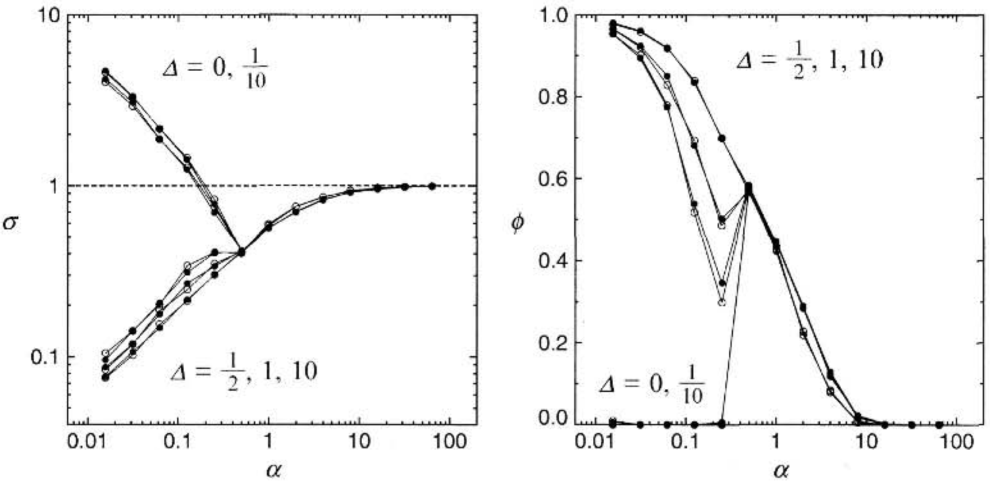

Fig. 1.9 Simulation measurements of the volatility σ (left) and the fraction ϕ of frozen agents (right). Here we compare the curves found for true market history(full circles, data identical to those in Fig. 1.7) to the corresponding measurements for 'fake' history (open circles), with all other definitions, initial conditions, and control parameters in the MG remaining the same. We see that the differences between real and fake history are surprisingly small.

Our conclusion must therefore be that for low α there are multiple macroscopic states, from which the system picks the one to which it is confined by the initial conditions. MG does appear to have a genuine non-equilibrium phase transition[1] at some critical value α_c, and is no longer ergodic for $\alpha < \alpha_c$.

1.3.5 Real versus fake memory

A final and peculiar (and above all mathematically helpful) piece of information surfaced when, it seems just out of curiosity, a modification of the original MG rules (1.1)–(1.3) was proposed, whereby rather than the true market history $\{\text{sgn}[A(\ell - M)], \ldots, \text{sgn}[A(\ell - 1)]\}$ one provides all agents at step ℓ of the game with a random string of M binary numbers (i.e. 'fake' market history). This simplifies our equations considerably, and turns the process (1.1)–(1.3) into a Markovian one.[2] If we define $p = 2^M$ with $p = \alpha N$, and write the look-up tables $\mathbf{R}^{ia}$ as binary vectors $\mathbf{R}^{ia} = (R_1^{ia}, \ldots, R_p^{ia}) \in \{-1, 1\}^p$, we find for this 'fake history' MG version

$$p_{ia}(\ell + 1) = p_{ia}(\ell) - A(\ell)\ R^{ia}_{\mu(\ell)}, \tag{1.9}$$

[1] Strictly speaking, true phase transitions are found only when $N \to \infty$. For finite system sizes one does expect ergodicity to be restored, but only on timescales that diverge with N.

[2] Markovian processes are stochastic processes where the probability to find a microscopic state at step $\ell + 1$ depends only on the microscopic state at step ℓ, not on the states at earlier time steps.

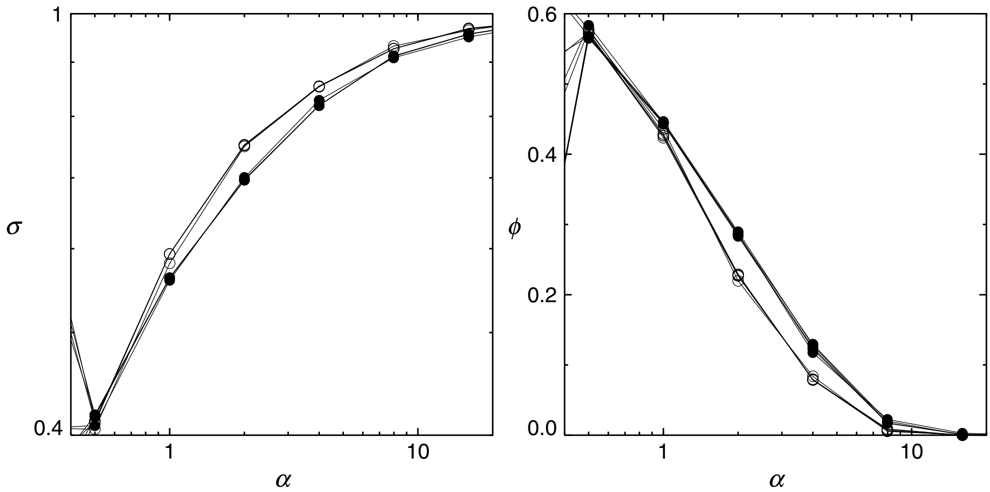

Fig. 1.10 Here we show close-ups in the large α region of the data in Fig. 1.9. Since these simulations are done in the ergodic region of the MG, where initial conditions are irrelevant, the variance in the data corresponding to $\Delta \in \{0,\ 0.1,\ 0.5,\ 1,\ 10\}$ in effect measures the error margins of our observations of σ and ϕ, for every value of α. We may conclude from these figures that for large α the macroscopic differences between real and fake memory, although small, are still real and non-negligible.

$$A(\ell) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} R_{\mu(\ell)}^{ia_i(\ell)}, \tag{1.10}$$

$$a_i(\ell) = \arg\max_{a \in \{1,\ldots,S\}} \{p_{ia}(\ell)\}. \tag{1.11}$$

Here the $\mu(\ell)$ are random numbers, drawn independently from $\{1, \ldots, p\}$ with equal probabilities, which define which of the $p = 2^M$ possible binary 'fake history' strings of length M one draws at the different steps ℓ.

At first sight, the consequences of this modification, see Figs 1.9 and 1.10, appear to leave little of the earlier suggestion that the efficient intermediate α regime of the MG might reflect the agents developing true expertise and learning to predict the market on the basis of historical market data.

- Upon simulating the modified 'fake history' MG process one observes nearly the same curves for the volatility σ and the frozen fraction ϕ as those corresponding to the original MG, including the efficient regime with $\sigma < 1$, in spite of the fact that the agents now take their decisions on the basis of 'historical' market data, which are in fact pure nonsense.

It would thus seem that the macroscopic behaviour of the MG market is not due to accumulating expertise of agents, but reflects other complicated collective processes.

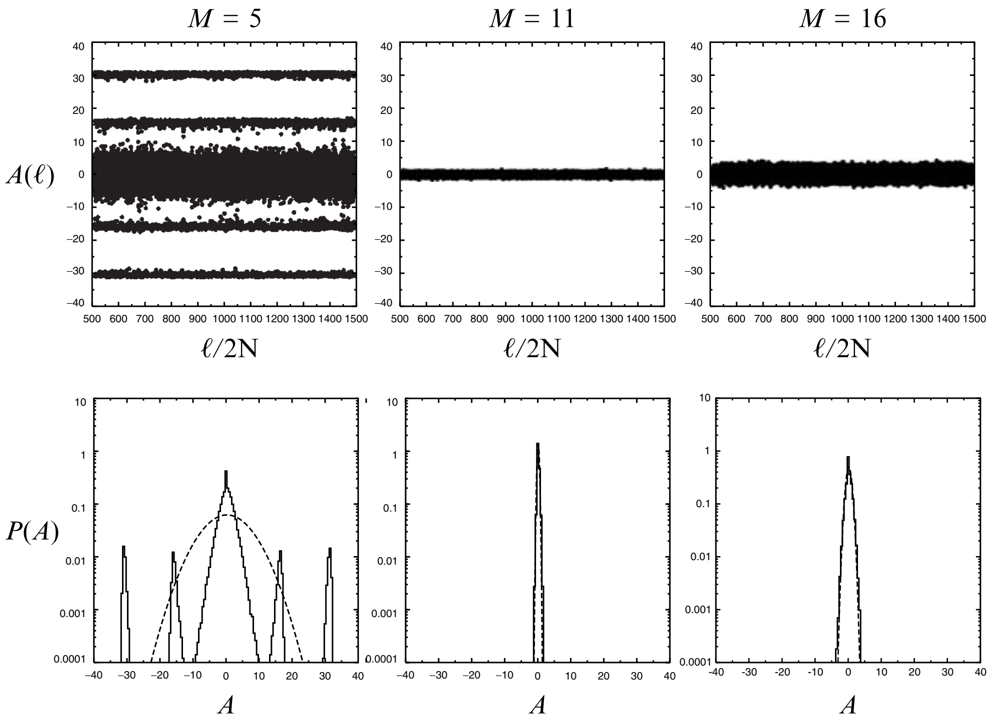

Fig. 1.11 Top row: simulation measurements of the re-scaled total bids $A(\ell)$, as observed under exactly the same conditions as in Fig. 1.2, but now with true market history replaced by 'fake' (i.e. random) data. Bottom row: the corresponding distributions $P(A)$ of these observed values (represented as histograms, and plotted logarithmically), together with Gaussian distributions (dashed) of width and average identical to those of the observed $P(A)$.

For an efficient market to emerge, i.e. in order to find $\sigma < 1$, it appears not to be required for the agents to develop any understanding of the dynamics of this market at all; it suffices that the agents themselves believe they do. The only essential ingredient is that the agents all act upon the same public data, be these true or nonsensical. Upon further reflection, however, this primitive view of the agents' role has to be moderated. Although the agents are obviously not learning to predict the time series of the overall bid (since that is indeed purely random), they do apparently learn to predict their *competitors' responses* to external information, irrespective of the quality of that information. This is the actual non-trivial 'understanding' of the market that the agents in the MG achieve in the intermediate α regime, and which is responsible for the low volatilities.

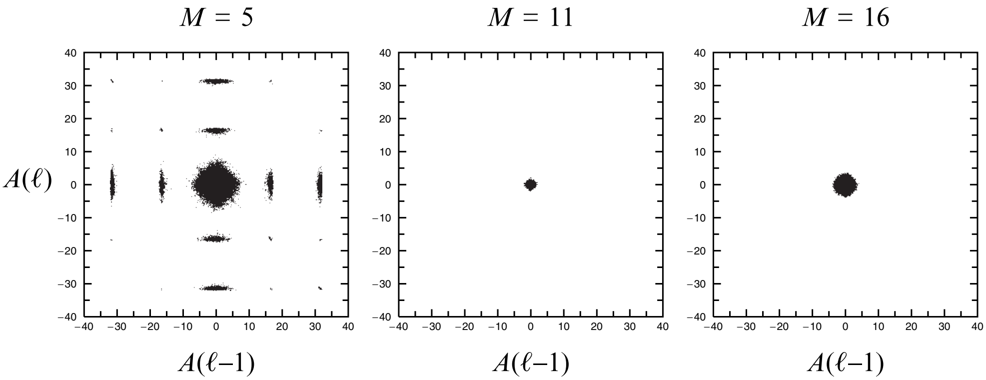

Fig. 1.12 Here we show the relation between the values of the re-scaled total bid at successive iteration steps in the simulations of the previous figure, in plots of $A(\ell)$ versus $A(\ell-1)$, but now with true market history replaced by 'fake' (i.e. random) data.

In the low α regime one observes that the differences between real and fake history, at least in terms of the values of σ and ϕ, are of the same order of magnitude as the accuracy of the simulations data. In the large α regime, in contrast (where, due to the system's ergodic behaviour, the different experimental curves obtained following different initial conditions Δ provide us with an accurate estimate of the reliability of the data), these differences are observed to be small but significant. See for instance Fig. 1.10 for close-ups of the simulation data in the large α regime. The question for which values of the MG's control parameters, if any, the real versus fake history scenarios, give truly (as opposed to nearly) identical results for the volatility σ and the fraction of frozen agents ϕ will be discussed in more mathematical detail later, in Chapter 8. For now, let us limit ourselves to collecting and showing also simulation data for the overall market bid $A(\ell)$ itself, by producing figures similar to our earlier Figs 1.2 and 1.3, but now for simulations of the 'fake memory' version of the MG. These figures suggest that for small α there are still no obvious differences between the true and fake history MG in the observed statistics $P(A)$ of the overall bid (for very small M and small N, however, this will certainly be different). When plotting the return maps of $A(\ell)$ versus $A(\ell-1)$ for small M, one does observe differences, with the fake history MG exhibiting more regular behaviour (as might have been expected).

At the mathematical level, however, replacing the original equations with true market history (1.1)–(1.3) by the 'fake' market history Markovian process (1.9)–(1.11) has significant implications. Markovian processes are in principle much easier to analyze mathematically than models with explicit memory, so that in the 'fake' history version of the MG we are much more likely to make progress using statistical mechanical techniques. In addition, in the fake history version of the MG we will no longer be restricted to choosing values for the main control parameter α which are of the form $\alpha = 2^M/N$ with M and N integers, but we can now take any value of the form $\alpha = p/N$ with p and N integers.

1.4 A preview

The rest of this book is devoted to the statistical mechanical analysis of the processes defined by the harmless looking equations (1.1)–(1.3) or (1.9)–(1.11), and some of their generalizations and variants, for large values of the system size N. To be specific: we will analyze our MG versions in the scaling regime where $N \to \infty$ for finite values for the ratio $\alpha = 2^M/N$ of the number of possible 'histories' divided by the number of agents. This is not a trivial exercise for mainly the following three reasons:

Obstacle 1: memory

The standard MG defines a non-Markovian process, due to the dependence of the microscopic laws on the global history of the market. The history parameter M (the number of time steps backward in time that influence any given iteration) will diverge as $N \to \infty$ (where N is the number of agents in the game), via $2^M = \alpha N$. Hence the history cannot be eliminated by a simple finite enlargement of the set of local microscopic variables.

Obstacle 2: frozen disorder

The MG contains frozen disorder, being the microscopic realization of the random look-up table entries $\{R_\mu^{ia}\}$ that define the agents' strategies. The total number of such random binary entries equals $\alpha S N^2$. We obviously cannot expect to be able to solve the MG equations for any given realization of this disorder.

Obstacle 3: no equilibrium

Unlike most physical systems, the microscopic laws of the MG do not obey detailed balance (i.e. the present model will exhibit microscopic probability currents in the stationary state), and hence the MG does not evolve to microscopic equilibrium. Worse still: we will find that the MG equations in fact have runaway solutions. This in principle rules out all the methods of equilibrium statistical mechanics; even in order to understand the stationary state, we have to solve the dynamics.

As we have seen, as a first step we may well replace the true market history in the original MG by (randomly and uniformly drawn) 'fake' market history, provided we accept in doing so that this modified Markovian version of the MG might behave slightly differently than the non-Markovian original. This replacement would at least eliminate obstacle one.[3] In the next chapter we will first prepare the stage for our statistical mechanical analyses, by deciding on the precise (generalized) MG definitions that we will mostly be working with, by inspecting in more detail the relevant time scales and the statistical properties of our MG processes, and by re-phrasing our equations in probabilistic language.

[3] In the same spirit some authors proposed a further modification of the MG in order to 'define away' also obstacle two (the frozen disorder), by replacing the independent random strategies by a more convenient set, which allows for the division of the agents into groups which have either effectively identical, anti-correlated, or un-correlated strategies. This, however, would not only take us too far from the original model, but it is also unnecessary, since (as we will see) appropriate mathematical tools are available to deal with the frozen disorder.

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2. Preparing the stage for statistical mechanics

We seem to have collected most of the information needed to launch a statistical mechanical analysis of the MG. We have established that the key control parameter in the MG should be $\alpha = 2^M/N$, and that for finite α we may expect the limit $N \to \infty$ to give well-defined statistics for the overall bids $A(\ell)$. We know which quantities to inspect, e.g. the volatility and the fraction of frozen agents. However, there remain issues that have not yet been resolved, such as the scaling with N of the characteristic timescales of the process (which do influence theory and simulations); these will be dealt with in the present chapter.

2.1 Generalized definition of MG

Before starting our calculations, however, it will be advantageous to generalize our definitions (1.1)–(1.3) slightly. Firstly, we would like to have a formulation which allows us to interpolate between the original equations (1.1)–(1.3) and their 'fake history' counterparts (1.9)–(1.11), to prevent unnecessary repetition of derivations later. Secondly, we would like to allow for the possibility that agents at times act somewhat irrationally, by not always selecting their best strategy (although on average they generally will). This appears more realistic, and will also turn out to produce interesting and at times counterintuitive effects. Finally, it will sometimes prove helpful in our analysis to complement the definition $A(\ell)$ of our overall bid by an external term $A_{\rm e}(\ell)$, reflecting e.g. external perturbations or actions by market regulators, and similarly to complement the agents' strategy valuations by external perturbation terms (which will allow us to define response functions later).

From now on we will employ the notation conventions that, unless stated otherwise, all summations over Roman indices will run over the set $\{1, \ldots, N\}$, all summations over $\boldsymbol{\lambda}$ will run over the set $\{-1, 1\}^M$ (where $2^M = \alpha N$), and all summations over Greek indices will run over the set $\{1, \ldots, p\}$ (where $p = \alpha N$).

2.1.1 Generalized MG without decision noise

As before, we imagine having N agents, labelled by $i = 1, \ldots, N$. At each iteration step $\ell \in \{0, 1, 2, \ldots\}$ of the game, each agent i submits a 'bid' $b_i(\ell) \in \{-1, 1\}$ to the market. The (re-scaled) cumulative market bid at stage ℓ is defined as

$$A(\ell) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} b_i(\ell) + A_{\mathrm{e}}(\ell). \tag{2.1}$$

Profit is assumed to be made by those agents who find themselves subsequently in the minority group, i.e. when $A(\ell) > 0$ by those agents i with $b_i(\ell) < 0$, and when $A(\ell) < 0$ by those with $b_i(\ell) > 0$. Each agent i determines his individual bid $b_i(\ell)$ at each step ℓ on the basis of publicly available external information, which the agents believe to represent historic market data, given by the vector $\boldsymbol{\lambda}(\ell, A, Z) \in \{-1, 1\}^M$:

$$\boldsymbol{\lambda}(\ell, A, Z) = \begin{pmatrix} \mathrm{sgn}[(1-\zeta)A(\ell-1) + \zeta Z(\ell, 1)] \\ \vdots \\ \mathrm{sgn}[(1-\zeta)A(\ell-M) + \zeta Z(\ell, M)] \end{pmatrix}. \tag{2.2}$$

The numbers $\{Z(\ell, \lambda)\}$, with $\lambda = 1, \ldots, M$, are zero-average Gaussian random variables, which represent a 'fake' alternative to the true market data. M is the number of iteration steps in the past for which market information is made available to the agents. We define $\alpha = 2^M/N$, and take α to remain finite as $N \to \infty$. The mixing parameter $\zeta \in [0, 1]$ allows us to interpolate between the extreme cases of strictly correct ($\zeta = 0$) and strictly fake ($\zeta = 1$) market information. We distinguish between two classes of 'fake history' variables

$$\text{consistent}: \; Z(\ell, \lambda) = Z(\ell - \lambda), \qquad \langle Z(\ell) Z(\ell') \rangle = \kappa^2 \delta_{\ell\ell'}, \tag{2.3}$$

$$\text{inconsistent}: \; Z(\ell, \lambda) \text{ all independent}, \quad \langle Z(\ell, \lambda) Z(\ell', \lambda') \rangle = \kappa^2 \delta_{\ell\ell'} \delta_{\lambda\lambda'}. \tag{2.4}$$

We note that equation (2.4) (which is the version actually used in producing e.g. Figs 1.9 and 1.11, with $\zeta = 1$) does not correspond to a consistent pattern being shifted in time, contrary to what one ought to expect of a string representing the time series of the overall bid, so that the agents in a real market could easily detect that they are being fooled. Hence equation (2.3) seems a more natural description of fake history. Although still fake, it is at least consistently so.

Each agent has S trading strategies, which we label by $a = 1, \ldots, S$. Each strategy a of each trader i consists of a complete list $\mathbf{R}^{ia}$ of 2^M recommended trading decisions $\{R^{ia}_{\boldsymbol{\lambda}}\} \in \{-1, 1\}$, covering all 2^M possible states of the external information vector $\boldsymbol{\lambda}$. We draw all entries $\{R^{ia}_{\boldsymbol{\lambda}}\}$ randomly and independently before the start of the game, with equal probabilities for ± 1. Upon observing history string $\boldsymbol{\lambda}(\ell, A, Z)$ at stage ℓ, given a trader's active strategy at that stage is $a_i(\ell)$, the agent will simply follow the appropriate instruction of his active strategy and take the decision $b_i(\ell) = R^{ia_i(\ell)}_{\boldsymbol{\lambda}(\ell,A,Z)}$. To determine their active strategies $a_i(\ell)$, all agents keep track of valuations $p_{ia}(\ell)$, which measure how often and to which extent each strategy a would have led to a minority decision if it had actually been used from the start of the game onwards. These valuations are updated continually, via[4]

$$p_{ia}(\ell+1) = p_{ia}(\ell) - \frac{\tilde{\eta}}{\sqrt{N}} A(\ell) R^{ia}_{\boldsymbol{\lambda}(\ell,A,Z)}. \tag{2.5}$$

The factor $\tilde{\eta}$ represents a learning rate. If the active strategy $a_i(\ell)$ of trader i at stage ℓ is defined as the one with the highest valuation $p_{ia}(\ell)$ at that point, and upon defining $\mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] = \sqrt{\alpha N}\; \delta_{\boldsymbol{\lambda}, \boldsymbol{\lambda}(\ell,A,Z)}$, our process can be written as

$$p_{ia}(\ell+1) = p_{ia}(\ell) - \frac{\tilde{\eta}}{N\sqrt{\alpha}} A(\ell) \sum_{\boldsymbol{\lambda}} R^{ia}_{\boldsymbol{\lambda}} \mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z], \tag{2.6}$$

$$A(\ell) = A_{\rm e}(\ell) + \frac{1}{N\sqrt{\alpha}} \sum_i \sum_{\boldsymbol{\lambda}} R^{ia_i(\ell)}_{\boldsymbol{\lambda}} \mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z], \tag{2.7}$$

$$a_i(\ell) = \arg \max_{a \in \{1,\ldots,S\}} \{p_{ia}(\ell)\}. \tag{2.8}$$

We note that $(\alpha N)^{-1} \sum_{\boldsymbol{\lambda}} 1 = (\alpha N)^{-1} \sum_{\boldsymbol{\lambda}} \mathcal{F}^2_{\boldsymbol{\lambda}}[\ell, A, Z] = 1$. We return to the MG versions that were studied numerically in the preceding section, upon putting $A_{\rm e}(\ell) \to 0$ (no external perturbations) and $\tilde{\eta} \to 1$. The standard MG is then recovered for $\zeta \to 0$ (i.e. true market data only), whereas the 'fake history' MG is found for $\zeta \to 1$ (i.e. fake market data only, of the inconsistent type (2.4)).

Two generalization are still to be implemented: adding time-dependent perturbations to the individual strategy valuations $\{p_{ia}(\ell)\}$, and the introduction of agents' decision noise. Both could have been done for an arbitrary S in the present section, but find somewhat less messy implementations for $S = 2$.

2.1.2 S=2, decision noise and valuation perturbations

Henceforth we will restrict ourselves mostly to the simplest case $S = 2$, where each agent has only two strategies, so $a \in \{1, 2\}$, since we have seen in Chapter 1 that

[4] In all our definitions so far the agents behave as 'price takers', i.e. they do not take into account their own impact on the global bid $A(\ell)$. In Chapter 7 we will inspect an MG version where this is no longer the case.

the choice made for S has only a quantitative effect on the behaviour of the MG. Our equations can now be simplified upon introducing the new variables

$$q_i(\ell) = \frac{1}{2}[p_{i1}(\ell) - p_{i2}(\ell)], \tag{2.9}$$

$$\boldsymbol{\omega}^i = \frac{1}{2}[\mathbf{R}^{i1} + \mathbf{R}^{i2}], \quad \boldsymbol{\xi}^i = \frac{1}{2}[\mathbf{R}^{i1} - \mathbf{R}^{i2}], \tag{2.10}$$

and $\boldsymbol{\Omega} = N^{-1/2}\sum_i \boldsymbol{\omega}^i$. Our previous rule $a_i(\ell) = \arg\ \max_a\{p_{ia}(\ell)\}$ for selecting active strategies reduces to $a_i(\ell) = 1$ if $q_i(\ell) > 0$ and $a_i(\ell) = 2$ if $q_i(\ell) < 0$ (with coin tossing to break ties). Thus, the bid of agent i at step ℓ is seen to follow from

$$\begin{aligned}\mathbf{R}^{ia_i(\ell)} &= \frac{1}{2}[\mathbf{R}^{i1} - \mathbf{R}^{i2}] + \frac{1}{2}\mathrm{sgn}[q_i(\ell)][\mathbf{R}^{i1} + \mathbf{R}^{i2}] \\ &= \boldsymbol{\omega}^i + \mathrm{sgn}[q_i(\ell)]\boldsymbol{\xi}^i.\end{aligned} \tag{2.11}$$

The above $S = 2$ formulation is easily generalized to include decision noise. We simply replace $\mathrm{sgn}[q_i(\ell)] \to \sigma[q_i(\ell), z_i(\ell)]$, in which the $\{z_j(\ell)\}$ are independent and zero-average random numbers, described by a symmetric and unit-variance distribution $P(z)$. The function $\sigma[q, z]$ is taken to be non-decreasing in q for any z, and parametrized by a control parameter $T \geq 0$ such that $\sigma[q, z] \in \{-1, 1\}$, with $\lim_{T\to 0}\sigma[q, z] = \mathrm{sgn}[q]$ and $\lim_{T\to\infty}\int \mathrm{d}z\ P(z)\sigma[q, z] = 0$. Typical examples are additive and multiplicative noise definitions such as

$$\text{Additive}: \ \sigma[q, z] = \mathrm{sgn}[q + Tz], \tag{2.12}$$

$$\text{Multiplicative}: \ \sigma[q, z] = \mathrm{sgn}[q]\ \mathrm{sgn}[1 + Tz]. \tag{2.13}$$

T measures the degree of randomness in the agents' decision making, with $T = 0$ bringing us back to $a_i(\ell) = \arg\ \max_a\{p_{ia}(\ell)\}$, and with purely random strategy selection for $T = \infty$. The distributions $P(z)$ chosen mostly in literature are

$$\text{Gaussian}: \ P(z) = (2\pi)^{-1/2}\mathrm{e}^{-1/2z^2} \tag{2.14}$$

$$\text{Boltzmann}: \ P(z) = \frac{1}{2}K[1 - \tanh^2(Kz)]. \tag{2.15}$$

The value of K in equation (2.15) follows from the requirement $\int \mathrm{d}z\ P(z)z^2 = 1$, giving $K = \{\frac{1}{2}\int \mathrm{d}z\ z^2[1 - \tanh^2(z)]\}^{1/2} \approx 0.907$. It turns out that in our theories we will in practice only need the averages

$$\sigma[q] = \int \mathrm{d}z\ P(z)\sigma[q, z]. \tag{2.16}$$

It follows from our definitions that the function (2.16) has the following properties: $\sigma[-q] = -\sigma[q]$, $\lim_{T\to 0}\sigma[q] = \mathrm{sgn}[q]$, $\lim_{T\to\infty}\sigma[q] = 0$, and $\partial\sigma[|q|]/\partial T < 0$. For additive noise (2.12) we also know that $\lim_{q\to\pm\infty}\sigma[q]=\pm 1$. For multiplicative noise (2.13) we find that $\sigma[q]$ will be of the form

$$\text{multiplicative}:\ \sigma[q]=\text{sgn}[q]\ \lambda(T),\quad \lambda(T)=\int \text{d}z\ P(z)\text{sgn}[1+Tz]. \tag{2.17}$$

Here $\lambda(T)\in[0,1]$, with $\lambda(0)=1$, $\lambda(\infty)=0$ and $\text{d}\lambda(T)/\text{d}T<0$ for all $T\geq 0$. For the four combinations defined by the standard choices (2.12), (2.13) and (2.14), (2.15) the integrations in equation (2.16) can be done without much difficulty, and the function $\sigma[q]$ is found to be

	Gaussian	Boltzmann
Additive	$\sigma[q]=\text{Erf}[\frac{q}{T\sqrt{2}}]$	$\sigma[q]=\tanh[\frac{Kq}{T}]$
Multiplicative	$\sigma[q]=\text{sgn}[q]\ \text{Erf}[\frac{1}{T\sqrt{2}}]$	$\sigma[q]=\text{sgn}[q]\ \tanh[\frac{K}{T}]$

(2.18)

Upon translating our previous microscopic laws (2.6) and (2.7) into the new language of the valuation differences (2.9) for $S=2$, we find that now our MG equations close in terms of our new dynamical variables $\{q_i(\ell)\}$, so that the perturbation of valuations can be implemented simply by replacing $q_i(\ell)\to q_i(\ell)+\theta_i(\ell)$, with $\theta_i(\ell)\in\mathbb{R}$. Thus we arrive at the following closed equations, defining our final generalized $S=2$ MG process

$$q_i(\ell+1)=q_i(\ell)+\theta_i(\ell)-\frac{\tilde{\eta}}{N\sqrt{\alpha}}\sum_{\boldsymbol{\lambda}}\xi^i_{\boldsymbol{\lambda}}\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]A(\ell), \tag{2.19}$$

$$A(\ell)=A_{\text{e}}(\ell)+\frac{1}{\sqrt{\alpha N}}\sum_{\boldsymbol{\lambda}}\Big\{\Omega_{\boldsymbol{\lambda}}+\frac{1}{\sqrt{N}}\sum_i\sigma[q_i(\ell),z_i(\ell)]\xi^i_{\boldsymbol{\lambda}}\Big\}\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z], \tag{2.20}$$

$$\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]=\sqrt{\alpha N}\ \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)} \tag{2.21}$$

with the information vector $\boldsymbol{\lambda}\in\{-1,1\}^M$ as given by equation (2.2). The values of the overall bids $\{A(\ell),Z(\ell,k)\}$ for $\ell\leq 0$ and of the valuation differences $q_i(0)$ play the role of initial conditions.

2.2 Timescales and nature of microscopic fluctuations

In order to appreciate the scaling with N of the relevant timescales and of the microscopic fluctuations in the MG, we will now study in more detail the stochastic process for the simplest case, i.e. $S = 2$ and inconsistent fake history. Hence we put $\zeta = 1$ in equation (2.2), we choose fake history (2.4), and we remove the external perturbations $\{\theta_i(\ell)\}$ and $\{A_e(\ell)\}$ from the process (2.19)–(2.21). Let us also write the state vector and the vector of decision noise variables at step ℓ as $\mathbf{q}(\ell) = (q_1(\ell), \ldots, q_N(\ell))$ and $\mathbf{z}(\ell) = (z_1(\ell), \ldots, z_N(\ell))$, respectively. Since for $\zeta = 1$ and for inconsistent fake history the vector $\boldsymbol{\lambda}(\ell, A, Z)$ will take each of its αN possible values randomly,[5] with equal probabilities, we make the following changes in our notation (with $p = \alpha N$):

$$\boldsymbol{\lambda} \in \{-1, 1\}^M \to \mu \in \{1, \ldots, p\}, \quad (\xi^i_{\boldsymbol{\lambda}}, \omega^i_{\boldsymbol{\lambda}}) \to (\xi^\mu_i, \omega^\mu_i), \quad \boldsymbol{\lambda}(\ell, A, Z) \to \mu(\ell)$$

with the random numbers $\mu(\ell)$ drawn independently from $\{1, \ldots, p\}$ at each step ℓ, with equal probabilities. We will also define $A^\mu[\mathbf{q}, \mathbf{z}]$ as the overall bid that would result when $\mu(\ell) = \mu$ and $\mathbf{z}(\ell) = \mathbf{z}$, and when the system state is $\mathbf{q}$. All this simplifies equations (2.19)–(2.21) to

$$q_i(\ell + 1) = q_i(\ell) - \frac{\tilde{\eta}}{\sqrt{N}} \xi_i^{\mu(\ell)} A^{\mu(\ell)}[\mathbf{q}(\ell), \mathbf{z}(\ell)], \tag{2.22}$$

$$A^\mu[\mathbf{q}, \mathbf{z}] = \Omega_\mu + \frac{1}{\sqrt{N}} \sum_i \sigma[q_i, z_i] \xi_i^\mu. \tag{2.23}$$

There are two sources of stochasticity in this Markovian process (2.22) and (2.23): the zero-average decision noise variables $z_i(\ell)$ with $\langle z_i(\ell) z_j(\ell') \rangle = \delta_{ij} \delta_{\ell\ell'}$, and the statistically independent 'fake history' integers $\mu(\ell) \in \{1, \ldots, p\}$.

2.2.1 Temporal regularization

It is straightforward to translate the process (2.22) and (2.23) into the language of microscopic probability densities $p_\ell(\mathbf{q})$, where $p_\ell(\mathbf{q})\mathrm{d}\mathbf{q}$ gives the probability of finding the state vector at stage ℓ of the game in an infinitesimal volume element $\mathrm{d}\mathbf{q} = \prod_i \mathrm{d}q_i$ located at point $\mathbf{q} \in \mathbb{R}^N$. The resulting equations involve a microscopic transition probability density operator $W(\mathbf{q}|\mathbf{q}') \geq 0$, which describes the averaging over the random numbers $\{z_i(\ell), \mu(\ell)\}$:

$$p_{\ell+1}(\mathbf{q}) = \int \mathrm{d}\mathbf{q}' \, W(\mathbf{q}|\mathbf{q}') p_\ell(\mathbf{q}'), \tag{2.24}$$

$$W(\mathbf{q}|\mathbf{q}') = \frac{1}{p} \sum_{\mu=1}^{p} \int \mathrm{d}\mathbf{z} \, P(\mathbf{z}) \prod_i \delta \left[q_i - q_i' + \frac{\tilde{\eta}}{\sqrt{N}} \xi_i^\mu A^\mu[\mathbf{q}', \mathbf{z}] \right] \tag{2.25}$$

with the short-hand $P(\mathbf{z}) = \prod_i P(z_i)$, and where $\delta[x]$ denotes the δ-function (see Appendix A). The quantity $W(\mathbf{q}|\mathbf{q}')\mathrm{d}\mathbf{q}$ gives the probability of finding the state vector

[5] Note that for consistent random history (2.4) and $\zeta = 1$ this would have been different: there would only be two allowed values for the history string $\boldsymbol{\lambda}(\ell, A, Z)$, the two options being determined by the previous history string $\boldsymbol{\lambda}(\ell - 1, A, Z)$.

in an infinitesimal volume element $\mathrm{d}\mathbf{q} = \prod_i \mathrm{d}q_i$ located at point $\mathbf{q} \in \mathbb{R}^N$, given that at the previous iteration step of the process (2.22) and (2.23) the state vector took the value $\mathbf{q}'$. The solution of equation (2.24) is

$$p_\ell(\mathbf{q}) = \int \mathrm{d}\mathbf{q}'\, W^\ell(\mathbf{q}|\mathbf{q}') p_0(\mathbf{q}'). \tag{2.26}$$

Here powers of the transition kernel $W(\mathbf{q}|\mathbf{q}')$ are defined in the usual manner, via $W^\ell(\mathbf{q}|\mathbf{q}') = \int \mathrm{d}\mathbf{q}''\, W^{\ell-1}(\mathbf{q}|\mathbf{q}'')W(\mathbf{q}''|\mathbf{q}')$.

Our next objective is to carry out temporal coarse-graining, and transform the present discrete-time dynamics to a new continuous-time process with an appropriately re-scaled continuous time t, which remains well defined in the limit $N \to \infty$. To do so we may use an established procedure for Markovian processes, and define the *duration* of each iteration to be a continuous random number, the statistics of which are described by the probability $\pi_\ell(t)$ that at time t precisely $\ell \geq 0$ updates have been made. Our new continuous-time process then becomes

$$p_t(\mathbf{q}) = \sum_{\ell \geq 0} \pi_\ell(t) p_\ell(\mathbf{q}) = \sum_{\ell \geq 0} \pi_\ell(t) \int \mathrm{d}\mathbf{q}'\, W^\ell(\mathbf{q}|\mathbf{q}') p_0(\mathbf{q}'). \tag{2.27}$$

For $\pi_\ell(t)$ one chooses the Poisson distribution $\pi_\ell(t) = \frac{1}{\ell!}(t/\Delta_N)^\ell \mathrm{e}^{-t/\Delta_N}$. Here Δ_N gives the average duration of an iteration step, and the relative uncertainty in the value of ℓ at a given $t \in \mathbb{R}$ follows from $\sqrt{\langle \ell^2 \rangle_\pi - \langle \ell \rangle_\pi^2}/\langle \ell \rangle_\pi = \sqrt{\Delta_N/t}$. The introduction of random step durations according to this procedure thus only introduces uncertainty about where we are on the time axis, which will vanish at the end of the calculation, provided $\lim_{N\to\infty} \Delta_N = 0$. Upon taking the time derivative of equation (2.27), we find that the convenient properties of the Poisson distribution under temporal derivation, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\pi_{\ell+1}(t) = \Delta_N^{-1}[\pi_\ell(t) - \pi_{\ell+1}(t)]$ and $\frac{\mathrm{d}}{\mathrm{d}t}\pi_0(t) = -\Delta^{-1}\pi_0(t)$ lead us to an elegant and simple continuous-time process:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} p_t(\mathbf{q}) &= \sum_{\ell \geq 0} \frac{\mathrm{d}\pi_\ell(t)}{\mathrm{d}t} \int \mathrm{d}\mathbf{q}'\, W^\ell(\mathbf{q}|\mathbf{q}') p_0(\mathbf{q}') \\
&= -\frac{1}{\Delta_N}\pi_0(t) p_0(\mathbf{q}') + \frac{1}{\Delta_N} \sum_{\ell > 0} [\pi_{\ell-1}(t) - \pi_\ell(t)] \int \mathrm{d}\mathbf{q}'\, W^\ell(\mathbf{q}|\mathbf{q}') p_0(\mathbf{q}') \\
&= \frac{1}{\Delta_N} \left\{ \int \mathrm{d}\mathbf{q}'\, W(\mathbf{q}|\mathbf{q}') p_t(\mathbf{q}') - p_t(\mathbf{q}) \right\}.
\end{aligned} \tag{2.28}$$

The advantage of the above regularization procedure is that is delivers a continuous time formulation of our MG process for any value of N. The continuous time limit and the limit $N \to \infty$ have thus been disconnected.

2.2.2 Kramers–Moyal expansion and timescaling

It will prove advantageous to expand the so-called master equation (2.28) in powers of the learning rate $\tilde{\eta}$. This implies expanding the transition kernel $W(\mathbf{q}|\mathbf{q}')$ (2.25) in a Taylor series. Upon defining the short-hands $\boldsymbol{\xi}^\mu = (\xi_1^\mu, \ldots, \xi_N^\mu)$, this series becomes

$$
\begin{aligned}
W(\mathbf{q}|\mathbf{q}') &= \frac{1}{p}\sum_{\mu=1}^{p}\int \mathrm{d}\mathbf{z}\ P(\mathbf{z})\delta\left[\mathbf{q}-\mathbf{q}'+\frac{\tilde{\eta}}{\sqrt{N}}\boldsymbol{\xi}^\mu A^\mu[\mathbf{q}',\mathbf{z}]\right] \\
&= \frac{1}{p}\sum_{\mu=1}^{p}\int \mathrm{d}\mathbf{z}\ P(\mathbf{z}) \sum_{n_1,\ldots,n_N\geq 0} \frac{\partial^{n_1+\cdots+n_N}\delta\left[\mathbf{q}-\mathbf{q}'\right]}{\partial q_1^{n_1}\cdots\partial q_N^{n_N}} \prod_i \frac{((\tilde{\eta}/\sqrt{N})\xi_i^\mu A^\mu[\mathbf{q}',\mathbf{z}])^{n_i}}{n_i!} \\
&= \sum_{k\geq 0}\frac{\tilde{\eta}^k}{N^{k/2}} \sum_{n_1,\ldots,n_N\geq 0} \frac{\delta_{k,\sum_i n_i}}{n_1!\cdots n_N!}\frac{\partial^k\delta\left[\mathbf{q}-\mathbf{q}'\right]}{\partial q_1^{n_1}\cdots\partial q_N^{n_N}} \\
&\quad\times \frac{1}{p}\sum_{\mu=1}^{p}\int \mathrm{d}\mathbf{z}\ P(\mathbf{z})(A^\mu[\mathbf{q}',\mathbf{z}])^k \prod_i (\xi_i^\mu)^{n_i}.
\end{aligned}
\tag{2.29}
$$

We see that the $k=0$ term in the expansion (2.29) equals simply $\delta[\mathbf{q}-\mathbf{q}']$, so that insertion of equation (2.29) into the master equation (2.28) results in the following so-called Kramers–Moyal expansion

$$
\frac{\mathrm{d}}{\mathrm{d}t}p_t(\mathbf{q}) = \sum_{k\geq 1}\left[L_k\, p_t\right](\mathbf{q}) \tag{2.30}
$$

with the operators

$$
\begin{aligned}
\left[L_k\, p\right](\mathbf{q}) = \frac{\tilde{\eta}^k}{N^{k/2}} &\sum_{n_1,\ldots,n_N\geq 0} \frac{\delta_{k,\sum_i n_i}}{n_1!\cdots n_N!}\frac{\partial^k}{\partial q_1^{n_1}\cdots\partial q_N^{n_N}} \\
&\times\left\{p(\mathbf{q})\left[\frac{1}{\Delta_N p}\sum_{\mu=1}^{p}(\xi_1^\mu)^{n_1}\cdots(\xi_N^\mu)^{n_N}\int \mathrm{d}\mathbf{z}\ P(\mathbf{z})(A^\mu[\mathbf{q},\mathbf{z}])^k\right]\right\}.
\end{aligned}
\tag{2.31}
$$

If we integrate equation (2.30) over all $\{q_i\}$ except one (say: q_k), we similarly find the expansion for the corresponding single-particle process, with $p_t(q_k)$ denoting the marginal distribution $p_t(q_k) = \int \mathrm{d}q_1\cdots\mathrm{d}q_{k-1}\mathrm{d}q_{k+1}\cdots\mathrm{d}q_N\ p_t(\mathbf{q})$ and with $\langle\cdots\rangle|_{q_k}$ denoting an average over $p_t(q_1,\ldots,q_{k-1},q_{k+1},\ldots,q_N|q_k) = p_t(\mathbf{q})/p_t(q_k)$:

$$
\frac{\mathrm{d}}{\mathrm{d}t}p_t(q_k) = \frac{\tilde{\eta}}{p\Delta_N}\frac{\partial}{\partial q_k}\left\{p_t(q_k)\left[\frac{1}{\sqrt{N}}\sum_\mu \xi_k^\mu\int \mathrm{d}\mathbf{z}\ P(\mathbf{z})\langle A^\mu[\mathbf{q},\mathbf{z}]\rangle\Big|_{q_k}\right]\right\}
$$

$$+\frac{\tilde{\eta}^2}{2p\Delta_N}\frac{\partial^2}{\partial q_k^2}\left\{p(q_k)\left[\frac{1}{N}\sum_\mu(\xi_k^\mu)^2\int \mathrm{d}\mathbf{z}\; P(\mathbf{z})\langle(A^\mu[\mathbf{q},\mathbf{z}])^2\rangle\bigg|_{q_k}\right]\right\}$$
$$+\frac{\tilde{\eta}^3}{6p\Delta_N}\frac{\partial^3}{\partial q_k^3}\left\{p(q_k)\left[\frac{1}{N^{3/2}}\sum_\mu(\xi_k^\mu)^3\int \mathrm{d}\mathbf{z}\; P(\mathbf{z})\langle(A^\mu[\mathbf{q},\mathbf{z}])^3\rangle\bigg|_{q_k}\right]\right\}$$
$$+\mathcal{O}(\tilde{\eta}^4) \tag{2.32}$$

We note that the first line of expansion (2.32) is the so-called flow term, which more or less controls the location of the centre of the distribution $p_t(q_k)$, whereas the subsequent terms control the fluctuations. For short times, and for the initial conditions normally envisaged for the MG there will be no $\mathcal{O}(1)$ correlations between the quantities $\int \mathrm{d}\mathbf{z}\; P(\mathbf{z})\langle(A^\mu[\mathbf{q},\mathbf{z}])^r\rangle|_{q_k}$ and ξ_k^μ, so that

$$\sum_\mu(\xi_k^\mu)^r\int \mathrm{d}\mathbf{z}\; P(\mathbf{z})\langle(A^\mu[\mathbf{q},\mathbf{z}])^r\rangle\bigg|_{q_k}=\begin{cases}\mathcal{O}(\sqrt{N}) & \text{for } r \text{ odd}\\ \mathcal{O}(N) & \text{for } r \text{ even.}\end{cases} \tag{2.33}$$

We will choose these scaling properties as our general ansatz. We are then automatically led to the choice $\Delta_N=\mathcal{O}(N^{-1})$, since this immediately ensures that the first and the second line of equation (2.33) will be of order $\mathcal{O}(N^0)$ (and hence remain well-defined in the limit $N\to\infty$). The higher order terms of the expansion are seen to be of vanishing order in N. The choice $\Delta_N=\mathcal{O}(N^{-1})$ also ensures that $\lim_{N\to\infty}\Delta_N=0$, which was the requirement underlying our time regularization procedure.[6] In accordance with the accepted convention in literature, we will also *en passant* pick up some further distracting terms (to find shorter equations later) and define

$$\Delta_N=\tilde{\eta}/2\alpha N. \tag{2.34}$$

The single-particle expansion (2.32) thereby becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(q_k)=\frac{\partial}{\partial q_k}\left\{p_t(q_k)\left[\frac{2}{\sqrt{N}}\sum_\mu\xi_k^\mu\int \mathrm{d}\mathbf{z}\; P(\mathbf{z})\langle A^\mu[\mathbf{q},\mathbf{z}]\rangle\bigg|_{q_k}\right]\right\}$$
$$+\tilde{\eta}\frac{\partial^2}{\partial q_k^2}\left\{p(q_k)\left[\frac{1}{N}\sum_\mu(\xi_k^\mu)^2\int \mathrm{d}\mathbf{z}\; P(\mathbf{z})\langle(A^\mu[\mathbf{q},\mathbf{z}])^2\rangle\bigg|_{q_k}\right]\right\}$$
$$+\mathcal{O}\left(\frac{1}{N}\right). \tag{2.35}$$

[6] The alternative to equation (2.33), expected to become relevant for pathological initial conditions, would imply having $\mathcal{O}(N)$ scaling for both r even and r odd in equation (2.33). With $\Delta_N=\mathcal{O}(N^{-1})$ this would reveal itself in a diverging flow term, which can be understood to reflect the system eliminating structural predictability on a much faster timescale than accidental predictability.

Both flow and diffusion terms are of order $\mathcal{O}(N^0)$, and higher orders vanish as N^{-1}. The fluctuations of individual components q_k are therefore Gaussian in the limit $N \to \infty$.

Having determined the canonical scaling with N of the average duration of individual iterations of the MG, via equation (2.34), we now find the various terms (2.31) in the full N-particle Kramers–Moyal expansion (2.30) reducing to

$$[L_k p](\mathbf{q}) = \frac{2\tilde{\eta}^{k-1}}{N^{k/2}} \sum_{\mu} \sum_{n_1,\ldots,n_N \geq 0} \frac{\delta_{k,\sum_i n_i}}{n_1! \cdots n_N!} (\xi_1^\mu)^{n_1} \cdots (\xi_N^\mu)^{n_N} \frac{\partial^k}{\partial q_1^{n_1} \cdots \partial q_N^{n_N}} \times \left\{ p(\mathbf{q}) \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) (A^\mu[\mathbf{q},\mathbf{z}])^k \right\}. \tag{2.36}$$

We may write this, equivalently, in the alternative form

$$[L_k p]\,(\mathbf{q}) = \frac{2\tilde{\eta}^{k-1}}{k!} \sum_{\mu} \left(N^{-1/2} \sum_i \xi_i^\mu \frac{\partial}{\partial q_i} \right)^k \left\{ p(\mathbf{q}) \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) (A^\mu[\mathbf{q},\mathbf{z}])^k \right\}. \tag{2.37}$$

Although equation (2.35) tells us that for $N \to \infty$ the individual components of the state vector have Gaussian fluctuations, one finds that the cumulative effect of the $k > 2$ terms in the expansion (2.30) cannot simply be neglected, since also the *number* of components q_k diverges with N. This can be seen more clearly in (2.37) than in equation (2.36). If, for instance, we work out the evolution of averages of observables $f(\mathbf{q})$, by insertion of equation (2.30) and integration by parts, we find

$$\begin{aligned} \tilde{\eta}\frac{\mathrm{d}}{\mathrm{d}t}\langle f\rangle &= \tilde{\eta} \int \mathrm{d}\mathbf{q}\, f(\mathbf{q}) \frac{\mathrm{d}}{\mathrm{d}t} p_t(\mathbf{q}) = \tilde{\eta} \sum_{k\geq 1} \int \mathrm{d}\mathbf{q}\, f(\mathbf{q})\, [L_k\, p_t]\,(\mathbf{q}) \\ &= 2 \sum_{k\geq 1} \frac{\tilde{\eta}^k}{k!} \int \mathrm{d}\mathbf{q}\, f(\mathbf{q}) \sum_{\mu} \left(N^{-1/2} \sum_i \xi_i^\mu \frac{\partial}{\partial q_i} \right)^k \left\{ p(\mathbf{q}) \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) (A^\mu[\mathbf{q},\mathbf{z}])^k \right\} \\ &= 2 \sum_{k\geq 1} \frac{(-\tilde{\eta})^k}{k!} \sum_{\mu} \left\langle \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) (A^\mu[\mathbf{q},\mathbf{z}])^k \big(N^{-\frac{1}{2}} \sum_i \xi_i^\mu \frac{\partial}{\partial q_i} \big)^k f(\mathbf{q}) \right\rangle. \end{aligned} \tag{2.38}$$

For simple observables such as $f(\mathbf{q}) = N^{-1} \sum_i \zeta_i q_i^{2m}$ (with finite m) one has $(N^{-1/2} \sum_i \xi_i^\mu \frac{\partial}{\partial q_i})^k f(\mathbf{q}) = \mathcal{O}(N^{-k/2})$, and hence the $k > 2$ terms in our expansion will be of vanishing order. In contrast, for observables that are dominated by fluctuations (such as those in the MG), this need no longer be the case. Let, for instance, $f(\mathbf{q}) = N^{-1/2} \sum_i \zeta_i q_i^{2m+1} = \mathcal{O}(N^0)$. Now we find $(N^{-1/2} \sum_i \xi_i^\mu \frac{\partial}{\partial q_i})^k f(\mathbf{q}) = \mathcal{O}(N^{-(k-1)/2})$, and therefore also the $k = 3$ term could for $N \to \infty$ in principle still contribute to the process.

2.2.3 The Fokker–Planck truncation

If we concentrate on only the first two terms $k = 1$ and $k = 2$ of equation (2.36) for the N-agent process (2.30), and insert definition (2.23) of $A^\mu[\mathbf{q}, \mathbf{z}]$, we end up with a Fokker–Planck equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(\mathbf{q}) = \sum_i \frac{\partial}{\partial q_i}[F_i(\mathbf{q}) p_t(\mathbf{q})] + \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial q_i \partial q_j}[M_{ij}(\mathbf{q}) p_t(\mathbf{q})] + \cdots \quad (2.39)$$

with

$$\begin{aligned} \frac{1}{2} F_i(\mathbf{q}) &= \frac{1}{\sqrt{N}} \sum_\mu \xi_i^\mu \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) \Big\{ \Omega_\mu + \frac{1}{\sqrt{N}} \sum_j \sigma[q_j, z_j] \xi_j^\mu \Big\} \\ &= \frac{\boldsymbol{\xi}_i \cdot \boldsymbol{\Omega}}{\sqrt{N}} + \frac{1}{N} \sum_j (\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j) \sigma[q_j], \end{aligned} \quad (2.40)$$

$$\begin{aligned} \frac{1}{2} M_{ij}(\mathbf{q}) &= \frac{\tilde{\eta}}{N} \sum_\mu \xi_i^\mu \xi_j^\mu \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) \Big\{ \Omega_\mu + \frac{1}{\sqrt{N}} \sum_k \sigma[q_k, z_k] \xi_k^\mu \Big\}^2 \\ &= \frac{\tilde{\eta}}{N} \sum_\mu \xi_i^\mu \xi_j^\mu \Big\{ \Omega_\mu + \frac{1}{\sqrt{N}} \sum_k \xi_k^\mu \sigma[q_k] \Big\}^2 \\ &\quad + \frac{\tilde{\eta}}{N^2} \sum_\mu \xi_i^\mu \xi_j^\mu \sum_k (\xi_k^\mu)^2 \{1 - \sigma^2[q_k]\} \end{aligned} \quad (2.41)$$

and with the function $\sigma[q]$ as defined in equation (2.16). In our expression (2.41) for the diffusion matrix elements of the Fokker–Planck equation we recognize the two distinct sources of fluctuations in the MG: the second term, which vanishes for $T \to 0$ reflects mostly the decision noise in the process, whereas the first reflects the noise due to the random selection of the fake history variables $\mu(\ell)$.

We note that the two expressions (2.40) and (2.41) are exact, but that the issue of whether or not we will be allowed to truncate the Kramers–Moyal expressions after the first two terms in equation (2.39) has not yet been settled. We will prove in Chapter 5, using more advanced methods, that this is indeed allowed in the limit $N \to \infty$. Furthermore, since the diffusion matrix (2.41) is still dependent upon the microscopic variables $\mathbf{q}$, even the truncated process (2.39) remains quite non-trivial.

2.3 Overview of MG versions to be analyzed

We are now in a position to state explicitly which MG models will be studied in the rest of this book, and how they are related. They will increase steadily in complexity, reflecting our increasing confidence and ambition as we go along, dealing in stages with the three technical obstacles that we identified earlier: the disorder (the randomly drawn strategies), the non-equilibrium nature of the dynamics (no detailed balance in the stationary state), and the non-Markovian nature of the process (via the market history). Most of our MG models will have $S = 2$ (two strategies per agent), and all are variations on the two basic sets of formulas (2.19)–(2.21), where we allow for true market history, and (2.28) and (2.34), which are the continuous time limits of (2.22) and (2.23) (for fake history only). All models below will be analyzed in the limit $N \to \infty$, with $\alpha > 0$ fixed:

Chapter 3: *Fake history MG—stationary state(s) in absence of fluctuations*

Here we first study a simplified version of equations (2.28) and (2.34) where we neglect all fluctuations, i.e. we truncate the Kramers–Moyal expansion after the first term. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}q_i(t) = -\frac{2}{\sqrt{N}}\sum_{\mu=1}^{\alpha N}\xi_i^\mu \int \mathrm{d}\mathbf{z}\; P(\mathbf{z})A^\mu[\mathbf{q}(t),\mathbf{z}], \tag{2.42}$$

$$A^\mu[\mathbf{q},\mathbf{z}] = \Omega_\mu + \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sigma[q_i,z_i]\xi_i^\mu. \tag{2.43}$$

This simplified process has neither history nor fluctuations, but has retained the disorder. In Chapter 3 we carry out a pseudo-equilibrium analysis of the stationary state(s) of equations (2.42) and (2.43), based on replica theory. This is found to work only above the critical value of α, in the ergodic regime.

Chapter 4: *Fake history MG—deterministic discrete-time (batch) dynamics*

The next stage is to study the dynamics directly, initially again for a simple history-free MG version similar to equation (2.42) and with $A^\mu[\mathbf{q},\mathbf{z}]$ again given by equation (2.43), but now with discrete time and with the decision noise fluctuations included (i.e. we no longer average over the variables $\mathbf{z}$):

$$q_i(t+1) = q_i(t) - \frac{2}{\sqrt{N}}\sum_{\mu=1}^{\alpha N}\xi_i^\mu A^\mu[\mathbf{q}(t),\mathbf{z}(t)]. \tag{2.44}$$

The advantage of discrete time is that it allows us to introduce and apply the formalism of generating functional analysis, and to obtain exact dynamical solutions of our MG models (formulated for correlation and response functions, and in terms of a single 'effective agent') in the ergodic and in the non-ergodic regime, without as yet having to involve path integrals.

Chapter 5: *Fake history MG—stochastic continuous-time (on-line) dynamics*
The natural next stage is to return to the original MG with fake history, and solve the full stochastic dynamics (2.25), (2.28), and (2.34) with all fluctuations included, i.e. to return to the stochastic version of (2.42) and (2.43):

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(\mathbf{q}) = \frac{1}{\Delta_N}\Big\{\int \mathrm{d}\mathbf{q}'\, W(\mathbf{q}|\mathbf{q}')p_t(\mathbf{q}') - p_t(\mathbf{q})\Big\}, \tag{2.45}$$

$$W(\mathbf{q}|\mathbf{q}') = \frac{1}{\alpha N}\sum_{\mu=1}^{\alpha N}\int \mathrm{d}\mathbf{z}\; P(\mathbf{z})\prod_{i=1}^{N}\delta\left[q_i - q_i' + \frac{\tilde{\eta}}{\sqrt{N}}\xi_i^\mu A^\mu[\mathbf{q}',\mathbf{z}]\right], \tag{2.46}$$

with $\Delta_N = \tilde{\eta}/2\alpha N$. This continuous-time process includes randomness due to the fake history realizations and the decision noise. Its generating functional analysis, leading again to exact closed dynamical laws for correlation and response functions in terms of a single 'effective agent', will now involve path integrals. In the limit $\tilde{\eta} \to 0$, i.e. for zero learning rate (where all fluctuations will vanish), the process (2.45) and (2.46) reduces to (2.42) and (2.43).

Chapter 6: *A study of the overall bid statistics in the previous models*
This chapter rounds off the analysis of the more conventional MG versions, as studied in the previous chapters, by calculating the distributions $P(A)$ of the overall bids, and by inspecting their dependence on having finite sizes N or finite relaxation times in the process.

Chapter 7: *Fake history MG—batch models with new transition types*
Here we inspect two variations on the basic batch MG, which show phase transitions that are qualitatively different from those in the previous models. In the first we allow the agents to (partially) correct their strategy valuation updates for their own impact on the market: we replace $A(\ell) \to A(\ell) - \rho N^{-1/2}R_{\mu(\ell)}^{ia_i(\ell)}$. This gives, again with definition (2.43):

$$q_i(t+1) = q_i(t) - \frac{2}{\sqrt{N}}\sum_{\mu=1}^{\alpha N}\xi_i^\mu\Big\{A^\mu[\mathbf{q}(t),\mathbf{z}(t)] - \frac{\rho}{\sqrt{N}}\Big[\omega_i^\mu + \sigma[q_i(t),z_i(t)]\xi_i^\mu\Big]\Big\}. \tag{2.47}$$

In the second model we allow agents to play linear combinations of their strategies, rather than pick their best. Here one is led to a linear theory. If we subsequently restore non-linearity by imposing a spherical constraint on the microscopic variables, we obtain a so-called spherical batch MG:

$$[1+\lambda(t+1)]q_i(t+1) = q_i(t) - \frac{2}{\sqrt{N}}\sum_{\mu=1}^{\alpha N}\xi_i^\mu A^\mu[\mathbf{q}(t)], \tag{2.48}$$

$$A^\mu[\mathbf{q}] = \Omega_\mu + \frac{1}{\sqrt{N}}\sum_{i=1}^{N} q_i \xi_i^\mu. \tag{2.49}$$

There is no decision noise, and the only nonlinearity is in global constraints $\sum_i q_i^2(t) = Nr^2$ (for all t), which determine the Lagrange parameters $\lambda(t)$.

Chapter 8: *Real history MG—stochastic continuous time dynamics*

The most complex level in our hierarchy of analyses is a generalization of the original non-Markovian game, with decision noise. Here the external information fed to the agents interpolates between fake and real history:

$$q_i(t+\Delta_N) = q_i(t) - 2\Delta_N\sqrt{\alpha}\sum_{\boldsymbol{\lambda}} \xi_{\boldsymbol{\lambda}}^i \mathcal{F}_{\boldsymbol{\lambda}}[t,A,Z]A(t), \tag{2.50}$$

$$A(t) = A_{\rm e}(t) + \frac{1}{\sqrt{\alpha N}}\sum_{\boldsymbol{\lambda}}\Big\{\Omega_{\boldsymbol{\lambda}} + \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sigma[q_i(t), z_i(\ell)]\xi_{\boldsymbol{\lambda}}^i\Big\}\mathcal{F}_{\boldsymbol{\lambda}}[t,A,Z], \tag{2.51}$$

$$\mathcal{F}_{\boldsymbol{\lambda}}[t,A,Z] = \sqrt{\alpha N}\,\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(t,A,Z)}, \tag{2.52}$$

with the information vector $\boldsymbol{\lambda} \in \{-1,1\}^M$ (where $2^M = \alpha N$) as defined in equation (2.2), which is a controlled interpolation between real ($\zeta = 0$) and and fake ($\zeta = 1$) market history. Here one finds again closed equations, but now involving correlation and response functions relating to both an 'effective agent' and to the evolving global bids themselves. When $\zeta = 1$ and $A_{\rm e}(t) = 0$ for all t, this process reduces to the Markovian process (2.45) and(2.46).

Chapter 9: *Batch MG variations aimed at increasing economic realism*

Finally we discuss two MG versions that aim to bring the model closer to real markets by introducing agent diversity into the game. There are many such variations, but only a few have so far been solved mathematically. In the first model one incorporates 'trend followers' into the MG, which are a fraction of agents who aim to make *majority* rather than minority decisions, as well as non-uniform decision noise levels, which gives

$$q_i(t+1) = q_i(t) + \frac{2\varepsilon_i}{\sqrt{N}}\sum_{\mu=1}^{\alpha N}\xi_i^\mu A^\mu[\mathbf{q}(t),\mathbf{z}(t)], \tag{2.53}$$

$$A^\mu[\mathbf{q},\mathbf{z}] = \Omega_\mu + \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sigma[q_i, z_i|T_i]\xi_i^\mu, \tag{2.54}$$

where $\varepsilon_i = -1$ for the usual 'contrarians', and $\varepsilon_i = 1$ for trend followers, and T_i measures the decision noise level of agent i.

In the second model one relaxes the requirement that all agents must place a bid at all times, by distinguishing between 'producers' (who must indeed always trade) and 'speculators' (who only trade when there is a profit incentive). In its simplest form (where all agents have but one strategy, and without decision noise) one finds the following equations

$$q_i(t+1) = q_i(t) - \varepsilon_i - \frac{2}{\sqrt{N}} \sum_{\mu=1}^{\alpha N} R_i^\mu A^\mu[\mathbf{q}(t)], \tag{2.55}$$

$$A^\mu[\mathbf{q}] = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \theta[q_i] R_i^\mu. \tag{2.56}$$

Here the agents will trade only if $q_i > 0$, and the variables $\varepsilon_i \in \mathbb{R}$ control their individual appetites for risk (one chooses $\varepsilon_i = -\infty$ for producers, so that the latter will always trade).

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3. Pseudo-equilibrium replica analysis

The first serious attempt in literature at solving an MG model using statistical mechanical techniques dealt with the simplest case of having fake and inconsistent market histories, i.e. with $\zeta = 1$ in equation (2.2) and with equation (2.4) describing fake history statistics, two strategies per agent (i.e. $S = 2$), and additive decision noise (2.12). It employed a strategy that had in the past proved effective in the study of non-equilibrium neural network models. The idea is to find a Lyapunov function $H(\mathbf{q})$ for the MG process, with $\mathbf{q} = (q_1, \ldots, q_N)$, which would allow one to determine the stationary state properties via an analysis of the minima of $H(\mathbf{q})$. This is mathematically equivalent to the ground state analysis of an equilibrium statistical mechanical system, and hence allows one to use equilibrium statistical mechanical techniques after all. Although in order to apply this idea it was initially found necessary to neglect all fluctuations in MG, including those induced by the random histories, the resulting theory proved remarkably successful.

3.1 The Lyapunov function

3.1.1 Deterministic approximation of the process

Our starting point is the Fokker–Planck equation (2.39), i.e. we truncate the Kramers–Moyal expansion (2.30) after the first two terms. Physically, this means that we assume the weak non-Gaussian fluctuations in our process not to conspire to an $\mathcal{O}(1)$ effect in the limit $N \to \infty$:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(\mathbf{q}) = \sum_i \frac{\partial}{\partial q_i} \left\{ p_t(\mathbf{q}) \Big[\frac{\boldsymbol{\xi}_i \cdot \boldsymbol{\Omega}}{\sqrt{N}} + \frac{1}{N} \sum_j (\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j) \sigma[q_j] \Big] \right\}$$
$$+ \frac{1}{2} \tilde{\eta} \sum_{ij} \frac{\partial^2}{\partial q_i \partial q_j} \left\{ p_t(\mathbf{q}) \Big[\frac{1}{N} \sum_\mu \xi_i^\mu \xi_j^\mu \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) \{A^\mu[\mathbf{q}, \mathbf{z}]\}^2 \Big] \right\}. \quad (3.1)$$

Clearly, the MG's fluctuations scale as $\mathcal{O}(N^0)$ for $N \to \infty$. This follows already from the $\mathcal{O}(N^0)$ scaling of the diagonal elements of the diffusion term in equation (3.1), even if we were to remove the decision noise. Thus the fluctuations of our process will vanish only for $\tilde{\eta} \to 0$.

Suppose that, in a first attempt at gaining some insight and against our better judgement, we were to forget for now about exactness and simply neglect the fluctuations altogether? We would then retain only the first term in equation (3.1), which describes the following deterministic evolution of the variables $\mathbf{q}$:

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i = -2\Big\{ \frac{\boldsymbol{\xi}_i \cdot \boldsymbol{\Omega}}{\sqrt{N}} + \frac{1}{N} \sum_j (\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j) \sigma[q_j] \Big\} = -\frac{\partial}{\partial \sigma[q_i]} L(\mathbf{q}) \tag{3.2}$$

with, using the shorthand (2.23):

$$\begin{aligned} L(\mathbf{q}) &= \sum_\mu \left\{ \Omega_\mu + \frac{1}{\sqrt{N}} \sum_j \xi_j^\mu \sigma[q_j] \right\}^2 \\ &= \sum_\mu \Big\{ \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) A^\mu[\mathbf{q}, \mathbf{z}] \Big\}^2. \end{aligned} \tag{3.3}$$

Clearly $L(\mathbf{q}) \geq 0$, i.e. bounded from below. For most of this chapter we will restrict our discussion to the additive decision noise examples in equation (2.18), so that the function $L(\mathbf{q})$ will be also smooth and continuously differentiable. For multiplicative noise the situation is different; here some form of regularization will be required, to which we will turn at the end of this chapter.

From equation (3.2) it now follows, in combination with the inequality $\mathrm{d}\sigma[q]/\mathrm{d}q \geq 0$ (since monotonicity of $\sigma[q]$ was built into our decision noise definitions), that

$$\frac{\mathrm{d}}{\mathrm{d}t} L(\mathbf{q}) = \sum_i \sigma'[q_i] \frac{\partial L(\mathbf{q})}{\partial \sigma[q_i]} \frac{\mathrm{d}q_i}{\mathrm{d}t} = -\sum_i \sigma'[q_i] \left[\frac{\partial L(\mathbf{q})}{\partial \sigma[q_i]} \right]^2 \leq 0. \tag{3.4}$$

The function $L(\mathbf{q})$ is bounded from below, continuously differentiable (at least for additive decision noise as in equation (2.18)), and its value is monotonically non-increasing in time. It hence plays the role of Lyapunov function for the deterministic equations (3.2). We conclude that in the absence of fluctuations, i.e. for $\tilde{\eta} \to 0$, the system's evolution will be directed towards a (possibly local) minimum of the function $L(\mathbf{q})$, although not via strict gradient descent as would have been the case for the deterministic limit of Langevin-type equilibrium systems with continuous variables.

3.1.2 Interpretation of minima and frozen agents

Upon reflection, the inequality (3.4) together with definition (3.3) make perfect sense in the context of the MG: for $\tilde{\eta} \to 0$, i.e. absent fluctuations, the system aims to suppress the noise-averaged magnitude of the overall bid, averaged over all possible realizations of the fake information (i.e. of the variable μ). The lower bound $L(\mathbf{q}) = 0$ (the ideal situation) would be achieved if

$$\forall\mu \in \{1, \ldots, p\}: \quad \Omega_\mu + \frac{1}{\sqrt{N}} \sum_j \xi_j^\mu \sigma[q_j] = 0. \tag{3.5}$$

These represent $p = \alpha N$ disordered linear equations to be obeyed by N real variables $\{\sigma[q_i]\}$. For large α one therefore expects equation (3.5) to not have a solution. Further complications arise due to the constraints on the outcome of the function $\sigma[q]$, viz. $\sigma[q] \in [-1, 1]$, which could limit the potential for solving equation (3.5) even more. More generally, we can say that equation (3.4) predicts evolution towards an asymptotic stationary state, which is the solution of a minimization problem with inequality constraints, namely the conditions $\sigma[q_i] \in [-1, 1]$ for all i. This minimization problem is solved using the method of Lagrange, i.e. from

$$\forall i \in \{1, \ldots, N\}: \quad \frac{\partial}{\partial q_i} L(\mathbf{q}) = \Lambda_i \frac{\partial}{\partial q_i} \sigma[q_i] \tag{3.6}$$

in which the $\{\Lambda_i\}$ denote the Lagrange parameters associated with the N constraints. This, in turn, implies that the solution $\mathbf{q}$ must be one where the agents can be divided into two distinct groups G_A and G_B, characterized by whether their particular Lagrange parameter is non-zero (i.e. $\Lambda_i \neq 0$, so the solution has $\sigma[q_i] = \pm 1$) or zero (i.e. $\Lambda_i = 0$, so the solution has $-1 < \sigma[q_i] < 1$), respectively

$$G_A = \Big\{ i \Big| \, \boldsymbol{\xi}_i \cdot \Big[\boldsymbol{\Omega} + \frac{1}{\sqrt{N}} \sum_j \boldsymbol{\xi}_j \sigma[q_j] \Big] \neq 0 \Big\}, \tag{3.7}$$

$$G_B = \Big\{ i \Big| \, \boldsymbol{\xi}_i \cdot \Big[\boldsymbol{\Omega} + \frac{1}{\sqrt{N}} \sum_j \boldsymbol{\xi}_j \sigma[q_j] \Big] = 0 \Big\}. \tag{3.8}$$

Comparison with equation (3.2) and the definition (2.16) shows that group G_A must consist of agents that at the relevant minimum of $L(\mathbf{q})$ have $\mathrm{d}\sigma[q_i]/\mathrm{d}q_i = 0$ and therefore values of q_i that diverge linearly with time

$$\lim_{t\to\infty} \frac{q_i(t)}{t} = -2 \Big\{ \frac{\boldsymbol{\xi}_i \cdot \boldsymbol{\Omega}}{\sqrt{N}} + \frac{1}{N} \sum_j (\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j) \sigma_j \Big\} = v_i \neq 0 \tag{3.9}$$

with $\sigma_j = \lim_{t\to\infty} \sigma[v_j t] = \mathrm{sgn}[v_j]$. Such agents apparently become fully devoted to just one of their two strategies, in the course of time. With additive decision noise they will never again change their active strategy during the game, and group G_A must therefore be identified with the population of ‘frozen’ agents as observed in our simulations. Only when the size of group G_A vanishes, should we expect to see a state with $L(\mathbf{q}) = 0$, i.e. with $\sum_\mu \{\int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) A^\mu[\mathbf{q}, \mathbf{z}]\}^2 = 0$. According to equation (3.5) this ought to happen for small values of α, which is perfectly consistent with the data given in Figure 1.6.

3.1.3 Conversion to a statistical mechanical problem

Our brutal decision to throw out all fluctuations in the evolution of $\mathbf{q}$ has already proven surprisingly fruitful. Let us now attempt a quantitative analysis of the minima of the function $L(\mathbf{q})$. Since $L(\mathbf{q})$ depends on $\mathbf{q}$ only via the quantities $\{\sigma[q_i]\}$, it makes sense to switch to the variables $\sigma_i = \sigma[q_i] \in [-1, 1]$ explicitly. We will write $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$. We will also re-scale our Lyapunov function and define instead $H(\boldsymbol{\sigma}) = \alpha^{-1} L(\mathbf{q})$, to give us somewhat simpler equations later, so

$$H(\boldsymbol{\sigma}) = \frac{1}{\alpha} \sum_{\mu=1}^{\alpha N} \Big\{ \Omega_\mu + \frac{1}{\sqrt{N}} \sum_j \xi_j^\mu \sigma_j \Big\}^2, \quad \boldsymbol{\sigma} \in [-1, 1]^N. \tag{3.10}$$

Finding the minimum of a scalar function $H(\boldsymbol{\sigma})$, which is bounded from below, and its dependence on control parameters such as α, is a familiar type of problem in equilibrium statistical mechanics. There such calculations, which are equivalent to finding the so-called ground state of a physical system in thermal equilibrium, with Hamiltonian $H(\boldsymbol{\sigma})$ (which in physics specifies the dependence of the total energy of the system on the microscopic degrees of freedom $\boldsymbol{\sigma}$), are carried out with the help of the partition function Z and the free energy[7] F:

$$Z = \mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathrm{e}^{-\beta H(\boldsymbol{\sigma})} \qquad F = -\beta^{-1} \log Z. \tag{3.11}$$

Here the operation $\mathrm{Tr}_{\boldsymbol{\sigma}}$ (the 'trace') is defined as $\mathrm{Tr}_{\boldsymbol{\sigma}} = \int_{-1}^{1} \cdots \int_{-1}^{1} d\boldsymbol{\sigma}$, since for now we deal with additive noise only. Our Hamiltonian (3.3) scales as $H(\boldsymbol{\sigma}) = \mathcal{O}(N)$ for $N \to \infty$, as required in statistical mechanical calculations. We may now define the free energy per agent as $f_N = F/N$, from which one obtains

$$\begin{aligned} \lim_{\beta\to\infty} f_N &= -\lim_{\beta\to\infty} \frac{1}{\beta N} \log \big[\mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\big] = \min_{\boldsymbol{\sigma}\in[-1,1]^N} H(\boldsymbol{\sigma})/N \\ &= \min_{\mathbf{q}\in\mathbb{R}^N} \frac{1}{p} \sum_{\mu=1}^{p} \Big\{ \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) A^\mu[\mathbf{q}, \mathbf{z}] \Big\}^2. \end{aligned} \tag{3.12}$$

In the context of MGs we may obviously forget about any underlying similarities with calculations of physical systems, if we wish, and regard equation (3.12) simply

[7] Note that the variable β in equation (3.11) is not related to the inverse of the decision noise level T, in spite of notational similarities with the temperature T and its inverse $\beta = 1/T$ in conventional equilibrium calculations in physics.

as a mathematical trick that allows us to calculate *properties* of minima of functions without knowing the actual coordinates of these minima.

We now also see a second reason why it was appropriate to switch from the variables $\mathbf{q}$ to the variables $\boldsymbol{\sigma}$. Since the set $[-1,1]^N$ is finite and $H(\boldsymbol{\sigma})$ is bounded from below, the trace in equation (3.11) is finite. In contrast, had we continued working with our original variables $\mathbf{q}$, we would have found ourselves with the alternative partition function $Z = \int_{\mathbb{R}^N} \mathrm{d}\mathbf{q}\, \exp[-\frac{\beta}{\alpha}L(\mathbf{q})]$, which diverges due to the saturation of $L(\mathbf{q})$ as $|q_i| \to \infty$, requiring us to introduce unnecessary and distracting regularizations.

Our expressions still involve frozen disorder, namely the realization of the randomly drawn strategy look-up tables. To deal with this problem we focus in our analysis on disorder-averaged quantities; such averaging will henceforth be denoted by $\overline{[\ldots]}$. Thus for large N we concentrate on the evaluation of $\overline{f} = \lim_{N\to\infty} \overline{f_N}$, which can be written as

$$\begin{aligned}\overline{f} &= -\lim_{N\to\infty} (\beta N)^{-1}\, \overline{\log Z} \\ &= -\lim_{N\to\infty} (\beta N)^{-1}\, \overline{\log \left[\mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathrm{e}^{-(\beta/\alpha)\sum_\mu \{\Omega_\mu + (1/\sqrt{N})\sum_j \xi_j^\mu \sigma_j\}^2}\right]}. \end{aligned} \qquad (3.13)$$

Apart from pathological cases, in statistical mechanical models of the present type the free energy per particle (or agent) is normally found to be self-averaging in the limit $N \to \infty$, i.e. dependent only on the statistical properties of the disorder $\{\Omega_\mu, \xi_i^\mu\}$ rather than on its detailed microscopic realization. From equation (3.13) we can now obtain, provided the two limits $\beta \to \infty$ and $N \to \infty$ commute (which, again, experience with similar models suggests)

$$E = \lim_{\beta\to\infty} \overline{f} = \lim_{N\to\infty} \overline{\min_{\mathbf{q}\in\mathbb{R}^N} \frac{1}{p}\sum_\mu \left\{\int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) A^\mu[\mathbf{q},\mathbf{z}]\right\}^2}. \qquad (3.14)$$

In a nutshell, the power of the statistical mechanical approach is that it allows one to extract exact mathematical statements regarding macroscopic quantities without knowing the outcome of the underlying microscopic process: we will be able to calculate equation (3.14) and related quantities without solving equation (3.2).

3.2 Dealing with disorder: replica theory

Carrying out the disorder average in equation (3.13) is seen to require averaging the logarithm of the trace of an exponential function of the original disorder variables $\{R_\mu^{ia}\}$ (with $\mu = 1, \ldots, \alpha N$, $i = 1, \ldots, N$, and $a = 1, 2$), which at first sight would seem impossible to do analytically. The replica method resolves this technical problem by using the identity $\overline{\log Z} = \lim_{n\to 0} n^{-1} \log \overline{Z^n}$, which when inserted into equation (3.13) gives us

$$\overline{f} = - \lim_{N\to\infty} \lim_{n\to 0} \frac{1}{nN\beta} \log \overline{\left[\mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathrm{e}^{-(\beta/\alpha)\sum_\mu \{\Omega_\mu + (1/\sqrt{N})\sum_j \xi_j^\mu \sigma_j\}^2} \right]^n}. \quad (3.15)$$

At this stage one makes a number of somewhat tricky but very powerful moves: one (i) writes the nth power of Z as the product $\prod_{\alpha=1}^n Z$ (which strictly speaking would be allowed only for integer n, whereas here we need $n \to 0$), (ii) writes this latter product as a single n-fold trace (by introducing n copies $\boldsymbol{\sigma}^\alpha$ of the original state vector $\boldsymbol{\sigma}$), and (iii) interchanges the limits $N \to \infty$ and $n \to 0$. The result is[8]

$$\overline{f} = - \lim_{n\to 0} \frac{1}{n\beta} \lim_{N\to\infty} \frac{1}{N} \log \left[\mathrm{Tr}_{\boldsymbol{\sigma}^1} \cdots \mathrm{Tr}_{\boldsymbol{\sigma}^n}\, \overline{\mathrm{e}^{-(\beta/\alpha)\sum_{\alpha=1}^n \sum_\mu \{\Omega_\mu + (1/\sqrt{N})\sum_j \xi_j^\mu \sigma_j^\alpha\}^2}} \right]. \quad (3.16)$$

The point of this exercise is that in equation (3.16), in contrast to equation (3.15), we can carry out the disorder averages (which will now simply factorize over the statistically independent entries $\{R_\mu^{ia}\}$ of the look-up tables) and continue our calculation. Henceforth Greek indices $\{\alpha, \beta\}$ will be used to label the copies (or 'replicas') of the system, and run from 1 to n.

3.2.1 Calculation of the disorder-averaged free energy

Upon inserting into equation (3.16) the definition of Ω_μ, as given below (2.10), and upon linearizing the np squares in the exponential of equation (3.16) via Gaussian integrals, i.e. inserting $\mathrm{e}^{(1/2)z^2} = \int Dx\, \mathrm{e}^{xz}$ with the short-hand $Dx = (2\pi)^{-1/2}\mathrm{e}^{-(1/2)x^2}$, we obtain

$$\overline{f} = - \lim_{n\to 0} \frac{1}{n\beta} \lim_{N\to\infty} \frac{1}{N} \log \left\{ \int \prod_{\alpha,\mu} [Dx_\alpha^\mu] \right.$$

[8] This mathematical procedure regarding the variable n could be (and often is) phrased in terms of analytical continuation from integer to real values of n (using convexity properties of f to guarantee uniqueness). However, in practice it is only possible to do this analytical continuation carefully for trivial cases. Yet, since its conception, the replica method (including the interchanging of the $N \to \infty$ and $n \to 0$ limits) has been applied with remarkable success to many different problems, and has been shown by various mathematical means to be much more trustworthy than it was initially thought to be. A proper discussion of replica theory would unfortunately bring us too far away from the present subject (for this I refer to appropriate specialized literature). Furthermore, our main mathematical tool for solving MG in subsequent chapters will not involve replica theory. Finally, choosing the notation α (which is already in use to denote the ratio $2^M/N$) for labelling the product of partition functions seems illogical, but reflects our desire not to deviate from established conventions in literature.

$$\times \operatorname{Tr}_{\boldsymbol{\sigma}^1} \cdots \operatorname{Tr}_{\boldsymbol{\sigma}^n} \overline{\mathrm{e}^{\mathrm{i}(2\beta/p)^{1/2} \sum_\alpha \sum_\mu x_\alpha^\mu \sum_j [\omega_j^\mu + \xi_j^\mu \sigma_j^\alpha]}} \Big\}. \tag{3.17}$$

We now restore the definitions (2.10) of $\{\omega_i^\mu, \xi_i^\mu\}$, i.e. $\omega_i^\mu = \frac{1}{2}(R_\mu^{i1} + R_\mu^{i2})$ and $\xi_i^\mu = \frac{1}{2}(R_\mu^{i1} - R_\mu^{i2})$, and carry out the disorder average

$$\begin{aligned}
\Xi &= \overline{\mathrm{e}^{\mathrm{i}(2\beta/p)^{1/2} \sum_\alpha \sum_\mu x_\alpha^\mu \sum_j [\omega_j^\mu + \xi_j^\mu \sigma_j^\alpha]}} \\
&= \overline{\mathrm{e}^{\mathrm{i}(\beta/2p)^{1/2} \sum_j \sum_\mu R_\mu^{j1} \sum_\alpha x_\alpha^\mu (1+\sigma_j^\alpha)}} . \overline{\mathrm{e}^{\mathrm{i}(\beta/2p)^{1/2} \sum_j \sum_\mu R_\mu^{j2} \sum_\alpha x_\alpha^\mu (1-\sigma_j^\alpha)}} \\
&= \prod_{j\mu} \left\{ \cos\left[\left(\frac{\beta}{2p}\right)^{1/2} \sum_\alpha x_\alpha^\mu (1+\sigma_j^\alpha) \right] \cos\left[\left(\frac{\beta}{2p}\right)^{1/2} \sum_\alpha x_\alpha^\mu (1-\sigma_j^\alpha) \right] \right\} \\
&= \prod_{j\mu} \mathrm{e}^{-(\beta/4p)[\sum_\alpha x_\alpha^\mu (1+\sigma_j^\alpha)]^2 - (\beta/4p)[\sum_\alpha x_\alpha^\mu (1-\sigma_j^\alpha)]^2 + \mathcal{O}(N^{-2})} \\
&= \prod_\mu \mathrm{e}^{-(\beta/2\alpha) \sum_{\alpha\beta} x_\alpha^\mu x_\beta^\mu \left[1 + (1/N) \sum_j \sigma_j^\alpha \sigma_j^\beta\right] + \mathcal{O}(1/N)}.
\end{aligned} \tag{3.18}$$

The $\mathcal{O}(\frac{1}{N})$ finite size correction term in the exponent is seen not to survive the limit $N \to \infty$ in equation (3.17). Thus, with $\mathbf{x} = (x_1, \ldots, x_n)$, expression (3.17) becomes (see Appendix B for multi-variate Gaussian integrals)

$$\begin{aligned}
\overline{f} &= -\lim_{n\to 0} \frac{1}{n\beta} \lim_{N\to\infty} \frac{1}{N} \log \Big\{ \operatorname{Tr}_{\boldsymbol{\sigma}^1} \cdots \operatorname{Tr}_{\boldsymbol{\sigma}^n} \\
&\quad \times \left[\int \frac{\mathrm{d}\mathbf{x}}{(2\pi)^{n/2}} \, \mathrm{e}^{-(1/2) \sum_{\alpha\beta} x_\alpha x_\beta \left(\delta_{\alpha\beta} + (\beta/\alpha)[1 + (1/N) \sum_j \sigma_j^\alpha \sigma_j^\beta]\right)} \right]^p \Big\} \\
&= -\lim_{n\to 0} \frac{1}{n\beta} \lim_{N\to\infty} \frac{1}{N} \log \left\{ \operatorname{Tr}_{\boldsymbol{\sigma}^1} \ldots \operatorname{Tr}_{\boldsymbol{\sigma}^n} \mathrm{e}^{-(p/2) \log\det[\mathbf{I} + (\beta/\alpha) \mathbf{D}(\boldsymbol{\sigma})]} \right\}.
\end{aligned} \tag{3.19}$$

Here $\mathbf{I}$ and $\mathbf{D}(\boldsymbol{\sigma})$ denote, respectively, the identity matrix in $\mathbb{R}^n$, and the $n \times n$ matrix with the following entries

$$D_{\alpha\beta}(\boldsymbol{\sigma}) = 1 + \frac{1}{N} \sum_j \sigma_j^\alpha \sigma_j^\beta. \tag{3.20}$$

The only object in equation (3.19) which involves the microscopic variables $\{\sigma_i^\alpha\}$, in the relevant orders in N, is found to be the $n \times n$ matrix with elements $q_{\alpha\beta} = N^{-1} \sum_j \sigma_i^\alpha \sigma_j^\beta$ (the usual object to play the leading role in replica calculations). This matrix is isolated from expression (3.19) by inserting the following representation of unity

$$1 = \prod_{\alpha\beta} \int \mathrm{d}q_{\alpha\beta} \, \delta[q_{\alpha\beta} - \frac{1}{N} \sum_j \sigma_j^\alpha \sigma_j^\beta].$$

If we write the above δ-functions in integral representation, using $\delta[z] = \int (\mathrm{d}\hat{z}/2\pi) \, \mathrm{e}^{\mathrm{i}z\hat{z}}$ (thereby generating n^2 new integration variables, to be called $\hat{q}_{\alpha\beta}$), use boldface symbols for the $n \times n$ matrices $\mathbf{q} = \{q_{\alpha\beta}\}$ and $\hat{\mathbf{q}}_{\alpha\beta} = \{\hat{q}_{\alpha\beta}\}$, and finally write the $n \times n$ matrix with all entries equal to 1 as $\mathbf{E}$ (so $E_{\alpha\beta} = 1$ for all α, β) we obtain the following expression

$$\overline{f} = -\lim_{n\to 0}\frac{1}{n\beta}\lim_{N\to\infty}\frac{1}{N}\log\int\frac{\mathrm{d}\mathbf{q}\mathrm{d}\hat{\mathbf{q}}}{(2\pi/N)^{n^2}}\,\mathrm{e}^{N\left[\mathrm{i}\sum_{\alpha\beta}\hat{q}_{\alpha\beta}q_{\alpha\beta}-(1/2)\alpha\log\det[\mathbf{I}+(\beta/\alpha)(\mathbf{E}+\mathbf{q})]\right]}$$

$$\times\,\mathrm{Tr}_{\boldsymbol{\sigma}^1}\cdots\mathrm{Tr}_{\boldsymbol{\sigma}^n}\mathrm{e}^{-\mathrm{i}\sum_j\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\sigma_j^\alpha\sigma_j^\beta}$$

$$= -\lim_{n\to 0}\frac{1}{n\beta}\lim_{N\to\infty}\frac{1}{N}\log\int\mathrm{d}\mathbf{q}\mathrm{d}\hat{\mathbf{q}}\;\mathrm{e}^{N\Psi[\mathbf{q},\hat{\mathbf{q}}]+\mathcal{O}(\log N)} \tag{3.21}$$

with

$$\Psi[\mathbf{q},\hat{\mathbf{q}}] = i\sum_{\alpha\beta}\hat{q}_{\alpha\beta}q_{\alpha\beta} - \frac{1}{2}\alpha\log\det\left[\mathbf{I}+\frac{\beta}{\alpha}(\mathbf{E}+\mathbf{q})\right]$$
$$+\log\mathrm{Tr}_{\sigma_1,\ldots,\sigma_n}\mathrm{e}^{-\mathrm{i}\sum_{\alpha\beta}\hat{q}_{\alpha\beta}\sigma_\alpha\sigma_\beta}. \tag{3.22}$$

In the limit $N \to \infty$ the integral in equation (3.21) over the matrices $\mathbf{q}$ and $\hat{\mathbf{q}}$ can be evaluated by steepest descent[9] (see e.g. Appendix A for definition and origin of the saddle-point method), and we obtain an appealing result in which we just need to find the relevant extremum (saddle-point) of the function (3.22), and subsequently analyze the behaviour of this extremum for $n \to 0$:

$$\overline{f} = -\lim_{n\to 0}\frac{1}{n\beta}\mathrm{extr}_{\mathbf{q},\hat{\mathbf{q}}}\;\Psi[\mathbf{q},\hat{\mathbf{q}}]. \tag{3.23}$$

3.2.2 The replica symmetric ansatz—background

Although expression (3.23) looks very simple, the continuation of a saddle-point which is an $n \times n$ matrix to values where $n < 1$ (which we are here asked to carry out) is in fact highly non-trivial, and has given rise to an enormous body of litera-

[9] It is appropriate at this stage to make a few clarifying notes on technicalities. Firstly, we have arrived at the present stage of our calculation upon choosing a specific scaling with N of the dummy integration variables $\{x_\alpha^\mu\}$ and $\{\hat{q}_{\alpha\beta}\}$. We have applied the rule that those exponents within our control must scale in exactly the same way with N as those where we do not have a choice. Here the Hamiltonian $H(\mathbf{q})$ as N^1, hence must all exponents which were introduced via dummy variables. Deviation from this prescription is allowed mathematically, but would give us different equations. However, if one were to continue the calculation with the alternative choice of scaling with N, one would either find a trivial saddle-point, or diverging finite N corrections (this is easily checked in simple toy models). In both cases one would thus be forced to re-trace one's steps and amend one's choices. Secondly, in the steepest descent evaluation of the integrals over $\{\mathbf{q},\hat{\mathbf{q}}\}$ one will find an imaginary saddle-point $i\hat{\mathbf{q}}^\star$ for $\hat{\mathbf{q}}$. Since $\hat{\mathbf{q}}$ was real-valued this implies that first we have to shift the $\hat{q}_{\alpha\beta}$ integrations in the complex plane according to $\hat{q}_{\alpha\beta} \to \hat{q}_{\alpha\beta} + i\hat{q}^\star_{\alpha\beta}$, and confirm that the extra integration contour segments in this plane generated by the shift give zero contributions to the original integral over $\hat{\mathbf{q}}$. In practice this is often a nasty exercise and is done only if there are reasons for suspicion.

ture of which we can here only mention those results which are of relevance for the present calculation. To see how to proceed from stage (3.23), we first make a detour and return to stage (3.13). We define the (disorder-dependent) microscopic measure $p(\boldsymbol{\sigma}) = Z^{-1}\exp[-\beta H(\boldsymbol{\sigma})]$ and the associated averages $\langle G(\boldsymbol{\sigma})\rangle$:

$$\langle G[\boldsymbol{\sigma}]\rangle = \frac{\mathrm{Tr}_{\boldsymbol{\sigma}}\, G(\boldsymbol{\sigma})\, \mathrm{e}^{-\frac{\beta}{\alpha}\sum_\mu \left\{\Omega_\mu + \frac{1}{\sqrt{N}}\sum_j \xi_j^\mu \sigma_j\right\}^2}}{\mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathrm{e}^{-\frac{\beta}{\alpha}\sum_\mu \left\{\Omega_\mu + \frac{1}{\sqrt{N}}\sum_j \xi_j^\mu \sigma_j\right\}^2}}. \tag{3.24}$$

In physics this is the so-called Boltzmann measure corresponding to the free energy F in equation (3.11). We may now use the following version of the replica method, which is taylored towards the evaluation of disorder-averaged expectation values

$$\begin{aligned}
\overline{\langle G(\boldsymbol{\sigma})\rangle} &= \overline{\left[\frac{\mathrm{Tr}_{\boldsymbol{\sigma}}\, G(\boldsymbol{\sigma})\mathrm{e}^{-\beta H(\boldsymbol{\sigma})}}{\mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}}\right]} = \overline{\left[\frac{\mathrm{Tr}_{\boldsymbol{\sigma}}\, G(\boldsymbol{\sigma})\mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\left[\mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\right]^{n-1}}{\left[\mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\right]^n}\right]} \\
&= \lim_{n\to 0} \mathrm{Tr}_{\boldsymbol{\sigma}^1}\cdots \mathrm{Tr}_{\boldsymbol{\sigma}^n}\, \overline{G(\boldsymbol{\sigma}^1)\mathrm{e}^{-\beta\sum_{\eta=1}^n H(\boldsymbol{\sigma}^\eta)}} \\
&= \lim_{n\to 0} \frac{1}{n}\sum_{\alpha=1}^n \mathrm{Tr}_{\boldsymbol{\sigma}^1}\cdots \mathrm{Tr}_{\boldsymbol{\sigma}^n}\, \overline{G(\boldsymbol{\sigma}^\alpha)\mathrm{e}^{-\frac{\beta}{\alpha}\sum_{\eta=1}^n\sum_{\mu=1}^p \left\{\Omega_\mu + \frac{1}{\sqrt{N}}\sum_j \xi_j^\mu \sigma_j^\eta\right\}^2}}.
\end{aligned} \tag{3.25}$$

As with the first version of the replica method, we have again used an intermediate step, which strictly speaking is allowed only for integer n. We may also introduce a second copy of our system, $\boldsymbol{\sigma}' = (\sigma_1', \ldots, \sigma_N')$, with the same state probabilities $p(\boldsymbol{\sigma}')$ and the same realization of the frozen disorder. A simple generalization of equation (3.25) allows us to express also averages involving two such copies in replica language

$$\begin{aligned}
\overline{\langle G(\boldsymbol{\sigma}, \boldsymbol{\sigma}')\rangle} &= \overline{\left[\frac{\mathrm{Tr}_{\boldsymbol{\sigma}}\mathrm{Tr}_{\boldsymbol{\sigma}'}\, G(\boldsymbol{\sigma}, \boldsymbol{\sigma}')\mathrm{e}^{-\beta H(\boldsymbol{\sigma})-\beta H(\boldsymbol{\sigma}')}}{\left[\mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\right]^2}\right]} \\
&= \overline{\left[\frac{\mathrm{Tr}_{\boldsymbol{\sigma}}\mathrm{Tr}_{\boldsymbol{\sigma}'}\, G(\boldsymbol{\sigma}, \boldsymbol{\sigma}')\mathrm{e}^{-\beta H(\boldsymbol{\sigma})-\beta H(\boldsymbol{\sigma}'}\left[\mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\right]^{n-2}}{\left[\mathrm{Tr}_{\boldsymbol{\sigma}}\, \mathrm{e}^{-\beta H(\boldsymbol{\sigma})}\right]^n}\right]} \\
&= \lim_{n\to 0} \mathrm{Tr}_{\boldsymbol{\sigma}^1}\cdots \mathrm{Tr}_{\boldsymbol{\sigma}^n}\, \overline{G(\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)\mathrm{e}^{-\beta\sum_{\eta=1}^n H(\boldsymbol{\sigma}^\eta)}} \\
&= \lim_{n\to 0} \frac{1}{n(n-1)}\sum_{\gamma\neq\rho}^n \mathrm{Tr}_{\boldsymbol{\sigma}^1}\cdots \mathrm{Tr}_{\boldsymbol{\sigma}^n}
\end{aligned}$$

$$\overline{\times\, G(\boldsymbol{\sigma}^{\gamma}, \boldsymbol{\sigma}^{\rho}) \mathrm{e}^{-\frac{\beta}{\alpha}\sum_{\eta=1}^{n}\sum_{\mu=1}^{p}\{\Omega_\mu + \frac{1}{\sqrt{N}}\sum_j \xi_j^\mu \sigma_j^\eta\}^2}}. \tag{3.26}$$

We will now apply these general tools (3.25) and (3.26) to the calculation of the following two disorder-averaged quantities

$$\overline{P(Q)} = \lim_{N\to\infty} \overline{\langle\, \delta[Q - \frac{1}{N}\sum_i \sigma_i^2]\,\rangle}, \tag{3.27}$$

$$\overline{P(q)} = \lim_{N\to\infty} \overline{\langle\langle\, \delta[q - \frac{1}{N}\sum_i \sigma_i \sigma_i']\,\rangle\rangle}, \tag{3.28}$$

$\overline{P(Q)}$ represents the distribution of (re-scaled) lengths of the state vector $\boldsymbol{\sigma}$, and $\overline{P(q)}$ the distribution of overlaps between $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ when both system copies have the same statistics (3.24) and the same disorder. It can be shown that for large N and in the case where in this limit there is just one ergodic sector in the system, the fluctuations in quantities such as $\frac{1}{N}\sum_i \sigma_i^2$ and $\frac{1}{N}\sum_i \sigma_i\sigma_i'$ must vanish, and that their values must be self-averaging, i.e. depend only on the statistical properties of the disorder rather than on its microscopic realization. Hence one must have

$$\overline{P(Q)} = \delta\Big[Q - \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\sigma_i^2\rangle}\Big], \tag{3.29}$$

$$\overline{P(q)} = \delta\Big[q - \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\sigma_i\rangle^2}\Big]. \tag{3.30}$$

On the other hand, if we make the choices $G(\boldsymbol{\sigma}) = \delta[Q - \frac{1}{N}\sum_i \sigma_i^2]$ and $G(\boldsymbol{\sigma}, \boldsymbol{\sigma}') = \delta[q - \frac{1}{N}\sum_i \sigma_i\sigma_i']$ in our two expressions (3.25) and (3.26), respectively, and repeat the very same manipulations which led us earlier from equation (3.13) to equation (3.21),[10] we find

$$\overline{P(Q)} = \lim_{n\to 0}\frac{1}{n}\sum_{\alpha=1}^{n}\lim_{N\to\infty}\frac{\int \mathrm{d}\mathbf{q}\mathrm{d}\hat{\mathbf{q}}\;\delta[Q - q_{\alpha\alpha}]\mathrm{e}^{N\Psi[\mathbf{q},\hat{\mathbf{q}}]+\mathcal{O}(\log N)}}{\int \mathrm{d}\mathbf{q}\mathrm{d}\hat{\mathbf{q}}\;\mathrm{e}^{N\Psi[\mathbf{q},\hat{\mathbf{q}}]+\mathcal{O}(\log N)}}, \tag{3.31}$$

$$\overline{P(q)} = \lim_{n\to 0}\frac{1}{n(n-1)}\sum_{\gamma\neq\rho}^{n}\lim_{N\to\infty}\frac{\int \mathrm{d}\mathbf{q}\mathrm{d}\hat{\mathbf{q}}\;\delta[q - q_{\gamma\rho}]\mathrm{e}^{N\Psi[\mathbf{q},\hat{\mathbf{q}}]+\mathcal{O}(\log N)}}{\int \mathrm{d}\mathbf{q}\mathrm{d}\hat{\mathbf{q}}\;\mathrm{e}^{N\Psi[\mathbf{q},\hat{\mathbf{q}}]+\mathcal{O}(\log N)}}. \tag{3.32}$$

As before, both these integrals are for $N \to \infty$ and finite n dominated by the saddle-point of the function $\Psi[\mathbf{q}, \hat{\mathbf{q}}]$, so that the result can be written solely in terms of the values of $\{\mathbf{q}, \hat{\mathbf{q}}\}$ *at* the saddle-point, i.e. at the extremum of equation (3.23):

$$\overline{P(Q)} = \lim_{n\to 0}\frac{1}{n}\sum_{\alpha=1}^{n}\delta[Q - q_{\alpha\alpha}]\big|_{\text{saddle}}, \tag{3.33}$$

[10] We also use the general normalization identity $\int \mathrm{d}\mathbf{q}\mathrm{d}\hat{\mathbf{q}}\;\mathrm{e}^{N\Psi[\mathbf{q},\hat{\mathbf{q}}]+\mathcal{O}(\log N)} = 1$ which follows upon making the trivial choice $G(\boldsymbol{\sigma}) = 1$ in equation (3.25).

$$\overline{P(q)} = \lim_{n\to 0} \frac{1}{n(n-1)} \sum_{\gamma\neq\rho}^{n} \delta[q - q_{\gamma\rho}]|_{\text{saddle}}. \tag{3.34}$$

Upon combining equation (3.30) with equations (3.33) and (3.34) we must conclude that assuming ergodicity and self-averaging for $N \to \infty$ of quantities such as $\frac{1}{N}\sum_i \sigma_i^2$ and $\frac{1}{N}\sum_i \sigma_i\sigma_i'$ translates in replica language into the following (so-called replica symmetry, RS) ansatz for the structure of the matrix $\mathbf{q}$ at the saddle-point

$$q_{\alpha\beta} = \delta_{\alpha\beta}\, Q + (1 - \delta_{\alpha\beta})\, q \tag{3.35}$$

with the physical meaning of Q and q as prescribed by equation (3.30), i.e.

$$Q = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \overline{\langle\sigma_i^2\rangle}, \quad q = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \overline{\langle\sigma_i\rangle^2}. \tag{3.36}$$

3.2.3 The RS ansatz—consequences

The ergodic (or RS) ansatz (3.35) is the final ingredient that will enable us to finish our calculation. We can now diagonalize the matrix $\mathbf{D} = \mathbf{E} + \mathbf{q}$ in equation (3.22), whose elements become $D_{\alpha\beta} = 1 + q + \delta_{\alpha\beta}(Q - q)$ and whose eigenvectors $\mathbf{x}$ and eigenvalues Λ are easily seen to be

$$\begin{aligned} \mathbf{x} = (1,\ldots,1): &\quad \Lambda = Q - q + n(1+q), &\quad \text{multiplicity} &= 1 \\ \mathbf{x}\cdot(1,\ldots,1) = 0: &\quad \Lambda = Q - q, &\quad \text{multiplicity} &= n-1. \end{aligned}$$

Hence

$$\begin{aligned} \log\det\left[\mathbf{I} + \frac{\beta}{\alpha}\mathbf{D}\right] &= \log\left[1 + \frac{\beta}{\alpha}[Q - q + n(1+q)]\right] + (n-1)\log\left[1 + \frac{\beta}{\alpha}(Q-q)\right] \\ &= n\left\{\frac{\beta(1+q)}{\alpha + \beta(Q-q)} + \log[1 + \frac{\beta}{\alpha}(Q-q)]\right\} + \mathcal{O}(n^2). \end{aligned} \tag{3.37}$$

One can easily convince oneself that the structure (3.35) for $\mathbf{q}$ implies a similar structure for the conjugate matrix $\hat{\mathbf{q}}$ (via the coupled saddle-point equations), for which we now put, with a modest amount of foresight: $\hat{q}_{\alpha\beta} = \frac{1}{2}i\beta^2 R\delta_{\alpha\beta} + \frac{1}{2}i\beta^2 r(1 - \delta_{\alpha\beta})$. We insert also this latter RS ansatz for $\hat{\mathbf{q}}$, together with equation (3.35) and (3.37), into the function (3.22) to be extremized. In the result we can achieve factorization of the remaining traces in the last term by linearizing the quadratic exponent, to give us

$$\frac{1}{n}\Psi_{\text{RS}}[\ldots] = \frac{1}{2}\beta^2[qr - QR] - \frac{1}{2}\alpha\left\{\frac{\beta(1+q)}{\alpha + \beta(Q-q)} + \log[1 + \frac{\beta(Q-q)}{\alpha}]\right\}$$

$$
\begin{aligned}
&+\frac{1}{n}\log \mathrm{Tr}_{\sigma_1\cdots\sigma_n}\int Dx\ \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sum_\alpha \sigma_\alpha^2+\beta x\sqrt{r}\sum_\alpha\sigma_\alpha}+\mathcal{O}(n)\\
&=\frac{1}{2}\beta^2[qr-QR]-\frac{1}{2}\alpha\left\{\frac{\beta(1+q)}{\alpha+\beta(Q-q)}+\log\left[1+\frac{\beta(Q-q)}{\alpha}\right]\right\}\\
&+\frac{1}{n}\log\int Dx\ \left\{\mathrm{Tr}_\sigma\ \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sqrt{r}\sigma}\right\}^n+\mathcal{O}(n)\\
&=\frac{1}{2}\beta^2[qr-QR]-\frac{1}{2}\alpha\left\{\frac{\beta(1+q)}{\alpha+\beta(Q-q)}+\log\left[1+\frac{\beta(Q-q)}{\alpha}\right]\right\}\\
&+\int Dx\ \log\ \mathrm{Tr}_\sigma\ \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sqrt{r}\sigma}+\mathcal{O}(n).
\end{aligned}
\tag{3.38}
$$

Insertion of expression (3.38) into expression (3.23) then leads to a clean and replica-free expression for the disorder-averaged free energy per agent, in RS ansatz

$$
\overline{f}_{\mathrm{RS}}=\mathrm{extr}_{q,Q,r,R}\ f[q,Q,r,R],
\tag{3.39}
$$

$$
\begin{aligned}
f[q,Q,r,R]&=\frac{1}{2}\beta(QR-qr)+\frac{1}{2}\alpha\left[\frac{1+q}{\alpha+\beta(Q-q)}+\frac{1}{\beta}\log\left[1+\frac{\beta(Q-q)}{\alpha}\right]\right]\\
&-\frac{1}{\beta}\int Dx\ \log\ \mathrm{Tr}_\sigma\ \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}.
\end{aligned}
\tag{3.40}
$$

The extremization of the function $f[q,Q,r,R]$ gives us the following four coupled saddle-point equations:[11]

$$
Q=\int Dx\ \left[\frac{\mathrm{Tr}_\sigma\ \sigma^2\mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}}{\mathrm{Tr}_\sigma\ \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}}\right],
\tag{3.41}
$$

$$
q=\int Dx\ \left[\frac{\mathrm{Tr}_\sigma\ \sigma\ \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}}{\mathrm{Tr}_\sigma\ \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}}\right]^2,
\tag{3.42}
$$

$$
r=\frac{\alpha(1+q)}{[\alpha+\beta(Q-q)]^2},\qquad R=\frac{\alpha(1+2q-Q)-\alpha^2/\beta}{[\alpha+\beta(Q-q)]^2}.
\tag{3.43}
$$

We note that from equation (3.43) one also obtains the useful relation

$$
\beta(r-R)=\frac{\alpha}{\alpha+\beta(Q-q)}.
\tag{3.44}
$$

According to equation (3.14) we must solve the coupled equations (3.41)–(3.43), insert the outcome into equation (3.40), and take the limit $\beta\to\infty$ in the result.

[11] In contrast to more conventional statistical mechanical calculations, i.e. those without replicas, we cannot here assume that the relevant saddle-point of $f[q,Q,r,R]$ is a minimum, since in the space of $n\times n$ matrices $\mathbf{q}$ one will find curvature sign changes at $n=1$. The standard example to show this is the minimization of the simple function $g(\mathbf{q})=\sum_{\alpha\beta=1}^n q_{\alpha\beta}^2$, which in RS ansatz (3.35) reduces to $g_{\mathrm{RS}}(\mathbf{q})=nQ^2+n(n-1)q^2$. We see that at $n=1$ the saddle-point $(q,Q)=(0,0)$ turns from a minimum (for $n>1$) into a saddle (for $n<1$). In practice, an effective method to determine the nature of the physical saddle-point is by first inspecting the high-temperature (i.e. small β) regime, where the solution is usually known.

This immediately gives us the following relatively simple expression for the disorder-averaged ground state energy per agent E in the limit $N \to \infty$:

$$
\begin{aligned}
E &= \lim_{N\to\infty} \overline{\min_{\mathbf{q}\in\mathbb{R}^N} \frac{1}{p}\sum_{\mu=1}^{\alpha N}\left\{\int \mathrm{d}\mathbf{z}\; P(\mathbf{z})A^{\mu}[\mathbf{q},\mathbf{z}]\right\}^2} \\
&= \lim_{\beta\to\infty}\left\{\frac{1}{2}\beta(QR-qr)+\frac{1}{2}\alpha\left[\frac{1+q}{\alpha+\beta(Q-q)}+\frac{1}{\beta}\log\left[1+\frac{\beta(Q-q)}{\alpha}\right]\right]\right. \\
&\quad \left. -\frac{1}{\beta}\int Dx\;\log\;\mathrm{Tr}_{\sigma}\;\mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}\right\}.
\end{aligned}
\tag{3.45}
$$

3.3 Harvesting the ground state

The hard work has now been done. All the information on the ground state that is accessible by equilibrium statistical mechanical techniques is contained in expression (3.45), in combination with the saddle-point equations (3.41)–(3.43).

3.3.1 The limit $\beta \to \infty$

To take the remaining limit in equation (3.45) we need to know how the relevant solution $\{q, Q, r, R\}$ of the RS saddle-point equations (3.41)–(3.43) scales with β as $\beta \to \infty$. To obtain this information we return briefly to the Boltzmann measure (3.24). We note that if we were to add a simple generating term to our original Hamiltonian, of the following form

$$
H(\boldsymbol{\sigma}) \;\to\; H(\boldsymbol{\sigma})+\gamma\sum_i \sigma_i, \quad \gamma\in\mathbb{R}
\tag{3.46}
$$

(with γ small), then upon repeated differentiation of the disorder-averaged free energy per agent $\overline{f}$ with respect to γ one would obtain the following expressions, involving the so-called susceptibility χ:

$$
\begin{aligned}
\chi &= -\lim_{\gamma\to 0}\frac{\partial^2\overline{f}}{\partial\gamma^2} = \beta\lim_{N\to\infty}\frac{1}{N}\sum_{ij}\overline{\langle\sigma_i\sigma_j\rangle-\langle\sigma_i\rangle\langle\sigma_j\rangle} \\
&= \beta\lim_{\gamma\to 0}\lim_{N\to\infty}\frac{1}{N}\overline{\left\langle\sum_i[\sigma_i-\langle\sigma_i\rangle]\right\rangle^2}.
\end{aligned}
\tag{3.47}
$$

This quantity is clearly non-negative, and zero only if $\langle\sigma_i^2\rangle = \langle\sigma_i\rangle^2$ for all i, i.e. when the system is completely frozen. Inspection of our derivation of the saddle-point eqns and of the ground state energy (3.41)–(3.45) reveals that the modification (3.46) would only lead to the following replacement in all formulas that contain a trace over σ:

$$\mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}} \;\rightarrow\; \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}-\beta\gamma\sigma}. \tag{3.48}$$

Thus, for small γ we would find equation (3.40) being modified according to

$$f[q,Q,r,R] = f[q,Q,r,R]|_{\gamma=0} - \frac{1}{2}\beta\gamma^2 \int Dx \left\{ \frac{\mathrm{Tr}_\sigma\, \sigma^2 \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}}{\mathrm{Tr}_\sigma\, \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}} - \left[\frac{\mathrm{Tr}_\sigma\, \sigma\, \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}}{\mathrm{Tr}_\sigma\, \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}}\right]^2 \right\} + \mathcal{O}(\gamma^3). \tag{3.49}$$

From this, via $\chi = -\lim_{\gamma\to 0} \partial^2 \overline{f}/\partial\gamma^2$, follow the following two equivalent expressions for χ in RS ansatz

$$\chi = \beta(Q-q), \tag{3.50}$$

$$\chi = \frac{1}{\sqrt{r}} \int Dx\, x\, \frac{\mathrm{Tr}_\sigma\, \sigma\, \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}}{\mathrm{Tr}_\sigma\, \mathrm{e}^{\frac{1}{2}\beta^2(R-r)\sigma^2+\beta x\sigma\sqrt{r}}}. \tag{3.51}$$

A diverging susceptibility marks a phase transition in the system, since it implies that an *infinitesimal* persistent energy perturbation of the type (3.46) manages to induce a *finite* change in the value of the observable $m = \lim_{N\to\infty} N^{-1}\sum_i \overline{\langle\sigma_i\rangle}$ (the disorder-averaged magnetization, which is itself obtained as the first derivative with respect to γ of the free energy per agent, and is zero for $\gamma = 0$).

It is clear from the saddle-point equations (3.41) and (3.42) that $0 \leq q \leq Q \leq 1$. Upon assuming $0 \leq \chi < \infty$ in the limit $\beta \to \infty$ it now follows immediately from (3.43) and (3.44) that

$$\lim_{\beta\to\infty} r = \frac{\alpha(1+q)}{(\alpha+\chi)^2}, \quad \lim_{\beta\to\infty} \beta(r-R) = \frac{\alpha}{\alpha+\chi}. \tag{3.52}$$

In the leading two orders in β^{-1} we may now eliminate both Q and R from our equations via $Q = q + \chi/\beta + \cdots$ and $R = r - \alpha/\beta(\alpha+\chi) + \cdots$, and write our RS results purely in terms of the $\beta \to \infty$ limits of the observables q and χ:

$$q = \int Dx \lim_{\beta\to\infty} \left\{ \frac{\mathrm{Tr}_\sigma\, \sigma^2\, \mathrm{e}^{-(\alpha\beta/2(\alpha+\chi))(\sigma-x\sqrt{(1+q)/\alpha})^2}}{\mathrm{Tr}_\sigma\, \mathrm{e}^{-(\alpha\beta/2(\alpha+\chi))(\sigma-x\sqrt{(1+q)/\alpha})^2}} \right\}, \tag{3.53}$$

$$\chi = \frac{\alpha+\chi}{\sqrt{\alpha(1+q)}} \int Dx\, x \lim_{\beta\to\infty} \left\{ \frac{\mathrm{Tr}_\sigma\, \sigma\, \mathrm{e}^{-(\alpha\beta/2(\alpha+\chi))(\sigma-x\sqrt{(1+q)/\alpha})^2}}{\mathrm{Tr}_\sigma\, \mathrm{e}^{-(\alpha\beta/2(\alpha+\chi))(\sigma-x\sqrt{(1+q)/\alpha})^2}} \right\}, \tag{3.54}$$

$$E = \frac{(\alpha+\chi)(\alpha-1-q)+\alpha\chi(1+q)}{2(\alpha+\chi)^2} - \int Dx \lim_{\beta\to\infty}\left\{\frac{1}{\beta}\log\ \mathrm{Tr}_\sigma\ \mathrm{e}^{-(\alpha\beta/2(\alpha+\chi))(\sigma-x\sqrt{(1+q)/\alpha})^2}\right\}. \tag{3.55}$$

Clearly the above traces over σ are for $\beta\to\infty$ all dominated by the value for which the exponential term which they have in common is maximal (via Laplace's argument, which is just the simplest version of the saddle-point method). We write the value of $\sigma\in[-1,1]$ which minimizes the quadratic exponents in the above traces as

$$s(a) = \arg\min_{\sigma\in[-1,1]}(\sigma-a)^2 = \begin{cases} a & \text{for} \quad |a|\le 1 \\ \mathrm{sgn}[a] & \text{for} \quad |a|>1 \end{cases}. \tag{3.56}$$

This allows us to write the results of taking the remaining limits $\beta\to\infty$ in the above equations as

$$q = \int Dx\ s^2\left(\frac{x\sqrt{1+q}}{\sqrt{\alpha}}\right), \quad \chi = \frac{\alpha\int Dx\ s'\left(x\sqrt{1+q}/\sqrt{\alpha}\right)}{\alpha-\int Dx\ s'\left(x\sqrt{1+q}/\sqrt{\alpha}\right)}, \tag{3.57}$$

$$E = \frac{1}{2(\alpha+\chi)}\left\{\frac{(\alpha+\chi)(\alpha-1-q)+\alpha\chi(1+q)}{\alpha+\chi} + \alpha\int Dx\left[s\left(\frac{x\sqrt{1+q}}{\sqrt{\alpha}}\right)-\frac{x\sqrt{1+q}}{\sqrt{\alpha}}\right]^2\right\}. \tag{3.58}$$

Finally we compactify our equations with the shorthand $u=\sqrt{\alpha/2(1+q)}\ge 0$, and simplify our expression (3.58) for E by working out the square inside the Gaussian integral and by subsequently using the saddle-point equations for q and χ. This gives

$$q = \int Dx\ s^2\left(\frac{x}{u\sqrt{2}}\right), \quad \chi = \frac{\alpha\int Dx\ s'\left(x/u\sqrt{2}\right)}{\alpha-\int Dx\ s'\left(x/u\sqrt{2}\right)}, \tag{3.59}$$

$$E = \frac{\alpha^2(1+q)}{2(\alpha+\chi)^2}, \quad u^2 = \frac{\alpha}{2(1+q)}. \tag{3.60}$$

Clearly $E\ge 0$, as it should be. We also see that one has $E\to 0$, which we expect to happen at sufficiently low α, only if $\chi\to\infty$. Hence we anticipate that upon reducing α the lower bound of $L(\mathbf{q})$ can only be achieved following a phase transition.

3.3.2 The volatility and the fraction of frozen agents

In order to assess the accuracy of the present analysis, it would help if we could also extract a prediction for the volatility σ and the fraction ϕ of frozen agents, which could then be tested against our numerical simulation results. In this chapter we have randomly drawn fake histories, we have neglected all fluctuations (so we also need no longer worry about correlations between the randomly drawn history indices $\mu(\ell)$ and the microscopic variables $\mathbf{q}(\ell')$), and we have assumed self-averaging. Thus within the present approximations we may write the volatility in the infinite system size limit as

$$\begin{aligned}
\sigma^2 &= \lim_{N\to\infty} \frac{1}{\alpha N} \sum_{\mu} \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} \mathrm{d}t \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) \overline{A_\mu^2[\mathbf{q}(t), \mathbf{z}]} \\
&= \lim_{N\to\infty} \frac{1}{\alpha N} \sum_{\mu} \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} \mathrm{d}t\, \overline{\left[\int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) A^\mu[\mathbf{q}(t), \mathbf{z}]\right]^2} \\
&\quad + \lim_{L,N\to\infty} \frac{1}{\alpha N^2 L} \sum_{\mu} \sum_{\ell=1}^{L} \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) \overline{\left[\sum_i \xi_i^\mu \left(\sigma[q_i(\ell), z_i(\ell)] - \sigma[q_i(\ell)]\right)\right]^2} \\
&= E + \lim_{L,N\to\infty} \frac{1}{\alpha N^2 L} \sum_{\mu} \sum_{\ell=1}^{L} \sum_{ij} \\
&\quad \times \overline{\xi_i^\mu \xi_j^\mu \left[\int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) \sigma[q_i(\ell), z_i(\ell)] \sigma[q_j(\ell), z_j(\ell)] - \sigma[q_i(\ell)] \sigma[q_j(\ell)]\right]}. \qquad (3.61)
\end{aligned}$$

In the sum $\sum_{ij} \cdots$ we use the fact that the random variables $\{z_i(\ell)\}$ at different sites are independent. Hence we retain only those terms where $i = j$ and find

$$\begin{aligned}
\sigma^2 &= E + \lim_{L,N\to\infty} \frac{1}{\alpha N^2 L} \sum_{\mu} \sum_{\ell=1}^{L} \sum_{i} \overline{(\xi_i^\mu)^2 \left[1 - \sigma^2[q_i(\ell)]\right]} \\
&= E + \frac{1}{2} \left\{1 - \lim_{\beta\to\infty} \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \sigma_i^2 \rangle}\right\}. \qquad (3.62)
\end{aligned}$$

Comparison with the definition (3.36), and with the property $\lim_{\beta\to\infty}(Q - q) = 0$ (as established earlier) reveals that, within the assumptions and approximations of the present chapter, one has the simple relation

$$\sigma^2 = E + \frac{1}{2}(1 - q). \qquad (3.63)$$

In contrast to the volatility, it seems that an expression for the fraction of frozen agents ϕ requires again a further replica calculation. We had identified frozen agents earlier as those agents i for which $\lim_{t\to\infty} q_i(t)/t = v_i \neq 0$, and hence $\sigma_i = \pm 1$ in the stationary state. We may therefore define, with the Boltzmann measure (3.24) and with the step function $\theta[z<0]=0$ and $\theta[z>0]=1$:

$$\phi = \lim_{\epsilon\uparrow 1}\lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\theta[\sigma_i^2-\epsilon^2]\rangle}. \tag{3.64}$$

To calculate equation (3.64) we may once more apply the replica identity (3.25), here to the observable $G(\boldsymbol{\sigma})=\theta[\sigma_i^2-\epsilon^2]$, followed again by the familiar manipulations that we used to calculate the ground state and by taking the limit $N\to\infty$:

$$\begin{aligned}
\phi &= \lim_{\epsilon\uparrow 1}\lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\theta[\sigma_i^2-\epsilon^2]\rangle}\\
&= \lim_{\epsilon\uparrow 1}\int Dx \lim_{\beta\to\infty}\left\{\frac{\mathrm{Tr}_\sigma\,\theta[\sigma^2-\epsilon^2]\,\mathrm{e}^{-(\alpha\beta/2(\alpha+\chi))(\sigma-x\sqrt{(1+q)/\alpha})^2}}{\mathrm{Tr}_\sigma\,\mathrm{e}^{-(\alpha\beta/2(\alpha+\chi))(\sigma-x\sqrt{(1+q)/\alpha})^2}}\right\}\\
&= \lim_{\epsilon\uparrow 1}\int Dx\,\theta\left[s^2\left(\frac{x}{u\sqrt{2}}\right)-\epsilon^2\right],
\end{aligned} \tag{3.65}$$

with, as before, $u=\sqrt{\alpha/2(1+q)}$.

3.3.3 Final predictions for additive decision noise

We can now proceed to the finish by inserting the function $s(a)$ as defined in equation (3.56), viz. $s(a)=\mathrm{sgn}[a]+\theta[1-|a|].(a-\mathrm{sgn}[a])$. All the remaining integrals in equations (3.59) and (3.65) can be done, see Appendix B for details (upon choosing $\lambda=1$ in the expressions found there), which gives the more explicit results

$$q = 1+\frac{1-2u^2}{2u^2}\mathrm{Erf}[u]-\frac{1}{u\sqrt{\pi}}\mathrm{e}^{-u^2}, \quad u=\sqrt{\frac{\alpha}{2(1+q)}}, \tag{3.66}$$

$$\chi = \frac{\alpha\,\mathrm{Erf}[u]}{\alpha-\mathrm{Erf}[u]}, \quad E=\frac{\alpha^2(1+q)}{2(\alpha+\chi)^2}, \tag{3.67}$$

$$\sigma = \sqrt{\frac{\alpha^2(1+q)}{2(\alpha+\chi)^2}+\frac{1}{2}(1-q)}, \quad \phi = 1-\mathrm{Erf}[u]. \tag{3.68}$$

We first solve the transcendental equations (3.66) numerically, upon inserting the expression for u into the equation for q, giving us q as a function of α. The result is substituted into equations (3.67) and (3.68) to give the susceptibility χ, the ground state energy E, the volatility σ, and the fraction of frozen agents ϕ. We are only allowed to have solutions with $0\le\chi<\infty$. Hence the acceptable solutions are found in the regime $u\ge\mathrm{Erf}^{-1}[\alpha]$. We see that the susceptibility diverges, and hence a phase transition occurs, when $u = \text{Erf}^{-1}[\alpha]$. Inserting this specific condition into equation (3.66) leads us to a simple closed equation from which to solve the corresponding critical value α_{c}:

$$\alpha_{\text{c}} = \text{Erf}[u_{\text{c}}]: \quad \text{Erf}[u_{\text{c}}] = 2 - \frac{1}{u_{\text{c}}\sqrt{\pi}}\text{e}^{-u_{\text{c}}^2} \tag{3.69}$$

Numerical solution of equation (3.69) is found to give the value $\alpha_{\text{c}} \approx 0.33740$, which agrees remarkably well with our earlier simulations in Chapter 1. We also see that the phase transition (3.69) occurs exactly when $\alpha = 1 - \phi$. This condition has a transparent dimensional interpretation: the MG tries to find a solution of the zero energy equation (3.5), and is seen to succeed in doing so exactly at the point where the ratio of the number of free dynamical variables (the number $N(1-\phi)$ of agents that are not 'frozen') over the number αN of random constraints becomes exactly one.

Below the transition point α_{c} we would find $\chi < 0$, so that our calculations in principle no longer apply. If we nevertheless try to extract information from our saddle-point equations for $\alpha < \alpha_{\text{c}}$, the only physically acceptable continuations must be those where we maintain $\chi = \infty$ (which is the value at the transition point) and $E = 0$ (see our earlier discussion regarding the interpretation of the energy minimum, where we argued that E should be a non-decreasing function of α). Thus for $\alpha < \alpha_{\text{c}}$ we would predict $u = \text{Erf}^{-1}[\alpha]$. This in fact allows for two possible continuations, since we are left with two expressions for q:

$$\text{branch 1}: \quad q = \frac{\alpha}{2\left[\text{Erf}^{-1}[\alpha]\right]^2} + 1 - \alpha - \frac{\text{e}^{-[\text{Erf}^{-1}[\alpha]]^2}}{\sqrt{\pi}\text{Erf}^{-1}[\alpha]}, \tag{3.70}$$

$$\text{branch 2}: \quad q = \frac{\alpha}{2\left[\text{Erf}^{-1}[\alpha]\right]^2} - 1. \tag{3.71}$$

Expansion of these two expressions for $\alpha \to 0$ gives $q = 1 - (2/3)\alpha + \mathcal{O}(\alpha^2)$ for branch 1, but $q = 2/\alpha\pi + \mathcal{O}(\alpha^0)$ for branch 2. Since $q \leq 1$ by definition, we see that the only candidate continuation for $\alpha < \alpha_{\text{c}}$ is expression (3.70).

Before showing the predictions of the present theory in full, upon solving the various coupled saddle-point equations (3.66) numerically, including formulas (3.67) and (3.68) for the susceptibility, the ground state energy, the volatility and the fraction of frozen agents, let us quickly check what the present theory predicts in the limits $\alpha \to \infty$ and $\alpha \to 0$. For $\alpha \to \infty$ we find

$$\lim_{\alpha\to\infty} q = 0, \quad \lim_{\alpha\to\infty} \chi = 1, \quad \lim_{\alpha\to\infty} E = \frac{1}{2}, \tag{3.72}$$

$$\lim_{\alpha\to\infty} \sigma = 1, \quad \lim_{\alpha\to\infty} \phi = 0. \tag{3.73}$$

These predictions are seen to be in perfect agreement with our earlier simulation results in Chapter 1. For $\alpha \to 0$, in contrast, we find upon taking the limit $u \to 0$ in our expressions (on the basis of continuation (3.70)) that

$$\lim_{\alpha\to 0} q = 1, \quad \lim_{\alpha\to 0} \chi = \infty, \quad \lim_{\alpha\to 0} E = 0, \tag{3.74}$$

$$\lim_{\alpha\to 0} \sigma = 0, \quad \lim_{\alpha\to 0} \phi = 1 - \alpha. \tag{3.75}$$

It is already clear from equations (3.74) and (3.75) that for small α the present theory, based on neglecting fluctuations, does not generally work. Both the volatility σ and the fraction of frozen agents ϕ fail to reproduce the observed behaviour in the simulations shown in Chapter 1, except for very strongly biased initial strategy preferences, which do indeed lead to a state with low volatility, scaling as $\sigma = \mathcal{O}(\sqrt{\alpha})$ for $\alpha \to 0$.

Numerical solution of our coupled equations (3.66)–(3.68) leads to Figs 3.1 and 3.2. In the figures for q, σ, and ϕ we also show the corresponding simulation data, as measured under conditions identical to those of the fake history MG of Figure 1.9,

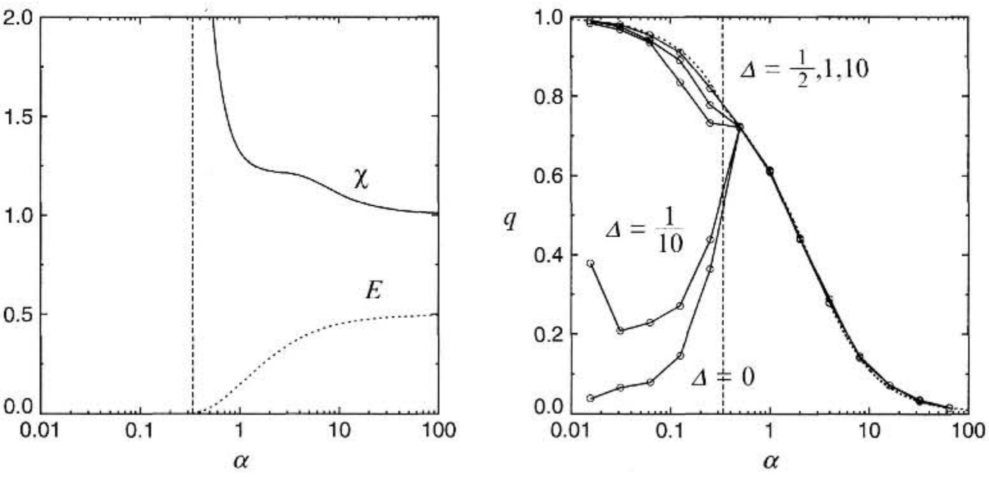

Fig. 3.1 Results of the pseudo-equilibrium replica symmetric theory, based on neglecting fluctuations and analyzing the ground state, for the 'fake history' MG with $S = 2$ and absent decision noise. Left: the predictions for the susceptibility χ (upper solid curve), and the disorder-averaged ground state energy per agent E (dotted curve). Right: the predictions for $q = N^{-1}\sum_i \overline{\langle\sigma_i\rangle^2}$ (dotted curve), together with the values observed in numerical simulations (connected markers) of the fake information MG of Figure 1.9. Initial conditions: $|q_i(0)| = \Delta$ for all i, with $\Delta \in \{0, 0.1, 0.5, 1, 10\}$ (from bottom to top in the region $\alpha < \alpha_c$). In both pictures the vertical dashed line marks the critical value $\alpha_c \approx 0.33740$ where the susceptibility χ diverges, marking a phase transition.

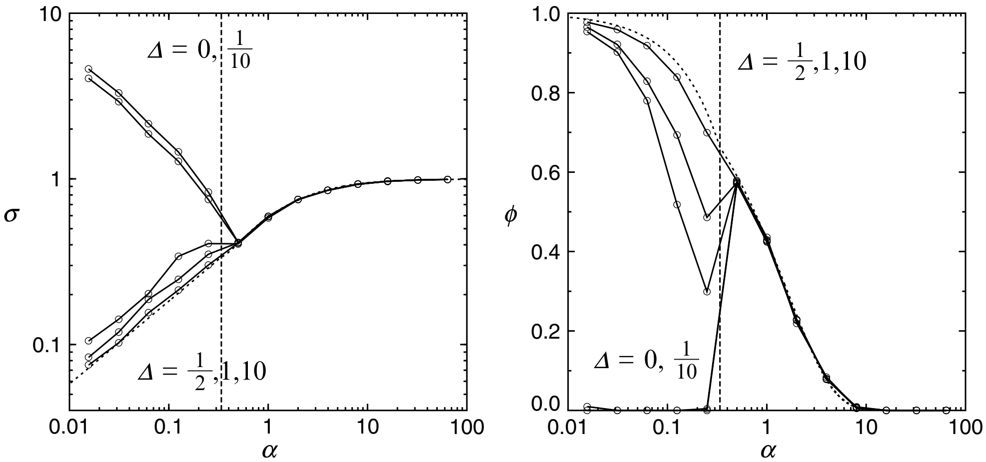

Fig. 3.2 Left: the predicted volatility σ (dotted curve), together with the simulation data (connected markers), for the fake history MG of figure 1.9 and absent decision noise. Right: the predicted fraction ϕ of frozen agents (dotted curve), together with the simulation data (connected markers). Initial conditions: $|q_i(0)| = \Delta$ for all i, with $\Delta \in \{0,\ 0.1,\ 0.5,\ 1,\ 10\}$ (from top to bottom in the region $\alpha < \alpha_c$ for σ; from bottom to top for ϕ). Again the vertical dashed line marks the critical value α_c.

for comparison (as yet without decision noise, so $T = 0$). Our figures show that the present crude approximation based on neglecting all fluctuations works amazingly well for $\alpha > \alpha_c$, and in fact appears to predict the location of the critical point quite accurately. It fails, however, to describe the correct phenomenology of the MG for small α (where our equations strictly speaking do not even apply). For $\alpha < \alpha_c$ the ad hoc continuation of our theory is found to single out only the low volatility solution corresponding to strongly biased initial strategy valuations.

Looking back at our final equations (3.66)–(3.68) we also note the rather surprising fact that none of these equations contains the noise parameter T. Apparently, the present theory predicts *en passant* that additive decision noise plays no role at the macroscopic level of quantities such as the present order parameters $\{q, \chi, \phi, E\}$. Numerical simulation experiments, see e.g. Figure 3.3, confirm that this rather counter-intuitive prediction is correct in the regime of the theory's validity (viz. for $\alpha > \alpha_c$) but that below α_c additive decision noise has in fact a drastic effect in bringing both the high volatility curves (emerging for 'tabula rasa' initial conditions) and the low volatility curves (emerging for significantly biased initial conditions) closer to the 'random decision making' benchmark value $\sigma = 1$. Since in the additive noise case the parameter T defines a natural scale for the variables q_i (see equation 2.18), we must conclude that at least above the phase transition point (i.e. for $\alpha > \alpha_c$) the physics

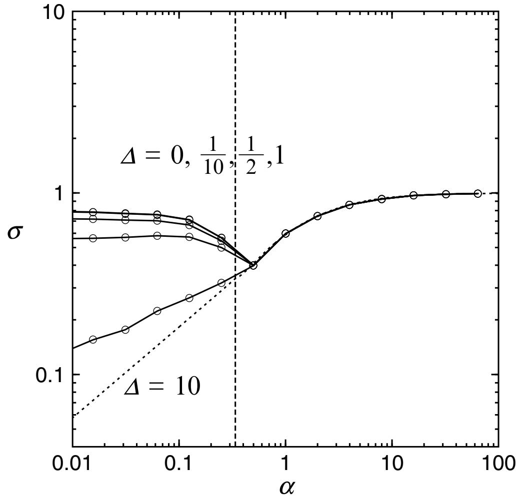

Fig. 3.3 The predicted volatility σ (dotted curve), together with the simulation data (connected markers), for the fake history MG with additive decision noise of strength $T = 1$. Initial conditions are as in the previous figure (the curves for $\Delta = 0$ and $\Delta = \frac{1}{10}$ are virtually indistinguishable). The differences between this case and that of Figure 3.2 is only in having switched on the (additive) decision noise, which according to the present theory should make no difference. This prediction is seen to be true only for $\alpha > \alpha_c$.

of the MG evolves only around the occurrence and multiplicity of the frozen agents, which indeed have diverging values of the $\{q_i\}$.

3.4 Predictions for multiplicative noise

The above results were all obtained for additive decision noise, where the function $\sigma[q]$ is smooth, and there are hence no singularities in equation (3.4). In the case of multiplicative noise (2.17), in contrast, we have $\sigma[q] = \lambda(T)\text{sgn}[q]$, so that $\sigma'[q] = 2\lambda(T)\delta[q]$ and our pseudo-equilibrium replica analysis requires some form of regularization. One way to do this is to replace $\sigma[q]$ temporarily by

$$\sigma[q] = \lambda(T)\tanh[kq], \tag{3.76}$$

where we must take $k \to \infty$ in our final result (although at this stage we do not yet know which of the limits $k \to \infty$, $\beta \to \infty$, and $N \to \infty$ commute).

3.4.1 Derivation of a phase diagram

With the choice (3.76) our previous analysis for additive decision noise requires only minor modifications. Once more we find the function $L(\mathbf{q})$ as given by equation (3.3) playing the role of Lyapunov function (3.4), and we may still use the equilibrium formalism based on equations (3.10) and (3.11), provided we now choose $\boldsymbol{\sigma} \in [-\lambda(T), \lambda(T)]^N$ and hence re-define the trace operation as $\mathrm{Tr}_{\boldsymbol{\sigma}} = \int_{-\lambda(T)}^{\lambda(T)} \cdots \int_{-\lambda(T)}^{\lambda(T)} \mathrm{d}\boldsymbol{\sigma}$. Thus we may take over in full detail the replica evaluation of the free energy per agent (3.16), which has indeed been written in the language of the trace (rather than in terms of the specific integrals appropriate for additive decision noise) for this very purpose. In fact we can move directly to equations (3.59) and (3.60), provided we re-define the function $s(a)$ in these expressions to

$$s(a) = \arg\min_{\sigma \in [-\lambda(T), \lambda(T)]} (\sigma - a)^2 = \begin{cases} a & \text{for } |a| \le \lambda(T) \\ \lambda(T).\mathrm{sgn}[a] & \text{for } |a| > \lambda(T) \end{cases}. \tag{3.77}$$

Careful inspection shows that also our earlier approximate expression (3.63) for the volatility remains valid, whereas the earlier formula (3.65) for the fraction of frozen agents is to be replaced by the new formula

$$\phi = \lim_{\epsilon \to 1} \int Dx \, \theta \left[s^2 \left(\frac{x}{u\sqrt{2}} \right) - \epsilon^2 \lambda^2(T) \right], \tag{3.78}$$

with, as before, $u = \sqrt{\alpha/2(1+q)}$. With multiplicative decision noise, however, one has to be careful in interpreting the observable ϕ. As with additive noise, ϕ equals the fraction of agents i for which $|q_i|$ diverges as time progresses, but it is no longer identical to the fraction of agents who never change strategy in the stationary state, since with multiplicative noise *every* agent i will continue to undergo such (random) changes, for any value of q_i. As a consequence ϕ can no longer be measured by counting strategy switches.

The details of the calculation of the three relevant integrals in the coupled order parameter equations can again be found in Appendix B. Now we choose $\lambda = \lambda(T)$ in the expressions in the appendix, which then gives us the following set of coupled equations

$$q = \lambda^2(T) \left\{ 1 + \frac{1 - 2v^2}{2v^2} \mathrm{Erf}[v] - \frac{1}{v\sqrt{\pi}} \mathrm{e}^{-v^2} \right\}, \tag{3.79}$$

$$v = \lambda(T) \sqrt{\frac{\alpha}{2(1+q)}}, \quad \chi = \frac{\alpha \, \mathrm{Erf}[v]}{\alpha - \mathrm{Erf}[v]}, \quad E = \frac{\alpha^2(1+q)}{2(\alpha+\chi)^2}, \tag{3.80}$$

$$\sigma = \sqrt{\frac{\alpha^2(1+q)}{2(\alpha+\chi)^2} + \frac{1}{2}(1-q)}, \quad \phi = 1 - \mathrm{Erf}[v]. \tag{3.81}$$

It is satisfactory to note that for $T \to 0$ (the limit of absent decision noise) we have $\lambda(0) = 1$, and we therefore return to the previous set of equations describing additive decision noise (which themselves did not depend on T). We also see that the regularization constant k in equation (3.76), which was to be sent to infinity at the end of the calculation, has in fact disappeared from our equations (similar to the vanishing of T from our additive decision noise theory).

As with additive decision noise, a phase transition occurs when the susceptibility diverges as we lower the value of α, and exactly at that point the disorder-averaged ground state energy E becomes zero. It is curious to observe that this happens (as before) precisely when $\alpha = 1 - \phi$, irrespective of the value of T, strengthening our earlier dimensional interpretation of the MG's phase transition. Here, however, in contrast to the situation of additive decision noise, the critical value for α will depend on the noise level T. It is now given by $\alpha_c(T) = \text{Erf}[v_c(T)]$, where $v_c(T)$ is the solution of the transcendental equation

$$\lambda^2(T)\left\{\text{Erf}[v] - 1 + \frac{1}{v\sqrt{\pi}}\text{e}^{-v^2}\right\} = 1. \tag{3.82}$$

The solution of equation (3.82) defines a transition line in the (α, T) plane, which separates a high α region with normal response (i.e. with $\chi < \infty$) from a low α region

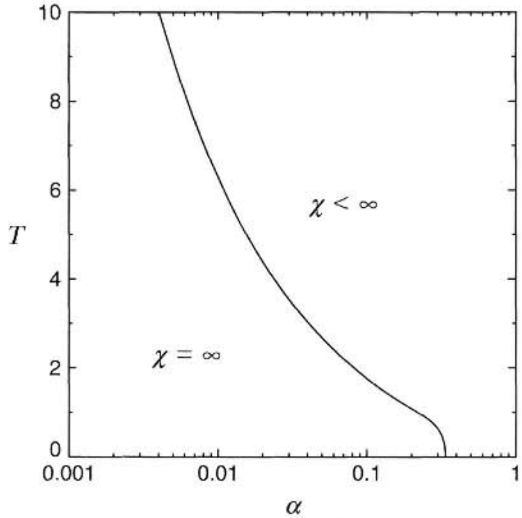

Fig. 3.4 Phase diagram in the (α, T) plane for the fake history MG with $S = 2$ and multiplicative decision noise, as predicted by the pseudo-equilibrium RS replica theory, which is based on neglecting fluctuations and analyzing the ground state. Here $\alpha = \lim_{N\to\infty} 2^M/N$ and T controls the noise strength. The transition line separates an ergodic phase with normal response ($\chi < \infty$) from a non-ergodic one with anomalous response ($\chi = \infty$).

with anomalous response (i.e. with $\chi = \infty$), giving rise to the phase diagram shown in Figure 3.4. At $T = 0$ (vanishing decision noise) we recover the earlier critical value $\alpha_c(T = 0) \approx 0.33740$, as we should.

3.4.2 Properties of the ground state

Below the transition point, where our present theory in principle no longer applies since $\chi < 0$ is forbidden by equation (3.47), we could try to continue the $\alpha > \alpha_c(T)$ solution similarly to how we proceeded for additive decision noise. This implies that for $\alpha < \alpha_c(T)$ we have to choose $\chi = \infty$, $E = 0$, $\lambda(T)u = \mathrm{Erf}^{-1}[\alpha]$, and

$$q = \lambda^2(T)\left\{\frac{\alpha}{2\left[\mathrm{Erf}^{-1}[\alpha]\right]^2} + 1 - \alpha - \frac{\mathrm{e}^{-[\mathrm{Erf}^{-1}[\alpha]]^2}}{\sqrt{\pi}\,\mathrm{Erf}^{-1}[\alpha]}\right\}. \tag{3.83}$$

The limit $\alpha \to \infty$ is found to come out exactly as for additive noise, for all our order parameters. For $\alpha \to 0$, the only difference with additive decision noise is in the limit found for q, here being $\lim_{\alpha\to 0} q = \lambda^2(T)$.

Numerical solution of the various equations derived in this section for multiplicative decision noise lead to Figs 3.5 and 3.6, here with the noise strength chosen to be

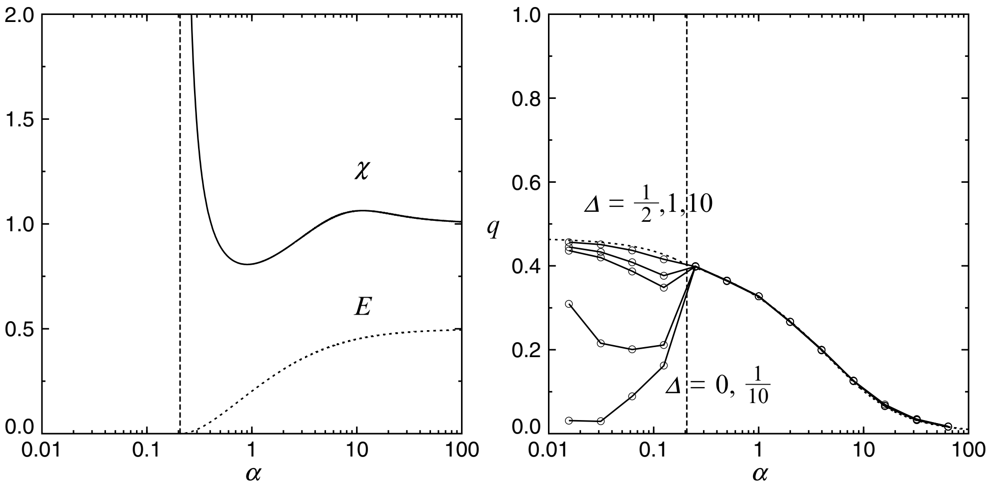

Fig. 3.5 Results of the pseudo-equilibrium RS replica theory for the fake history MG with $S = 2$ and multiplicative decision noise of strength $T = 1$. Left: the predictions for the susceptibility χ (upper solid curve) and the disordered-averaged ground state energy per agent E (dotted curve). Right: the predicted (dotted) and observed values (connected markers) for the order parameter q. Initial conditions: $|q_i(0)| = \Delta$ for all i, with $\Delta \in \{0,\ 0.1,\ 0.5,\ 1,\ 10\}$ (from bottom to top). The vertical dashed lines mark the critical value $\alpha_c(T = 1) \approx 0.20693$ where the susceptibility χ diverges.

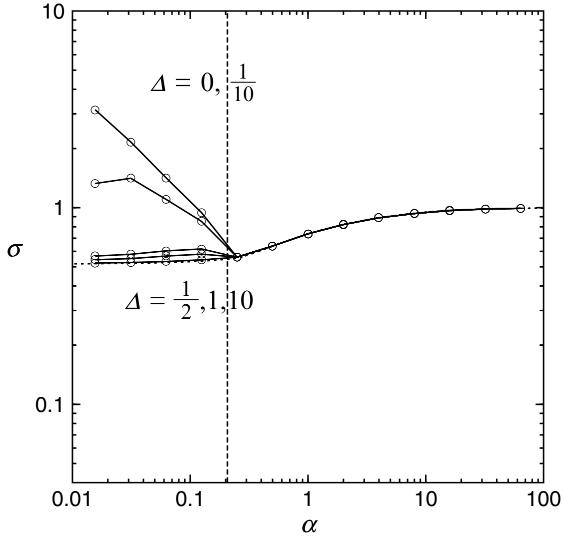

Fig. 3.6 Predictions of the pseudo-equilibrium RS theory for the volatility (dotted curve) in the fake history MG with multiplicative decision noise of strength $T = 1$, together with the corresponding simulation data (connected markers). Initial conditions: $|q_i(0)| = \Delta$ for all i, with $\Delta \in \{0,\ 0.1,\ 0.5,\ 1,\ 10\}$ (from top to bottom in the region $\alpha < \alpha_c$). The vertical dashed line marks the predicted critical value $\alpha_c(T = 1) \approx 0.20693$.

$T = 1$. In these figures we also show the corresponding numerical simulation data for the observables q and σ, for comparison. It is clear from our data that our crude approximation of neglecting all fluctuations again works amazingly well for $\alpha > \alpha_c$, and also once more appears to predict the location of the critical point quite accurately (including, it seems, the dependence of the critical value for α on the noise strength T), but again the theory is not capable of describing the correct MG phenomenology for small α. This is consistent with our knowledge that our equations are invalid below the critical point. For small α the theory again generates only the solution that emerges following strongly biased initial conditions, i.e. for large Δ. We also see that in both theory and experiment, in contrast to the case of additive decision noise, there will never be a solution with vanishing volatility for $\alpha \to 0$. This, of course, makes perfect sense: even if *all* agents were to have diverging values of $|q_i|$, for multiplicative noise they would all continue to undergo noise-induced changes in their active strategy, with direct implications for the global fluctuations in the system.

3.5 An assessment

It is difficult not to be impressed by the achievements of our pseudo-equilibrium replica analysis. The calculation was systematic and had the inevitability and autonomy of a tank engine in motion, which distinguishes a good theory from ad hoc patchwork: once the assumptions were made, the final predictions were already written in stone, including its failure to explain the high volatilities for $\alpha < \alpha_c$ (following 'tabula rasa' type initial conditions) and the effects of decision noise in the low α regime, its apparent accuracy for $\alpha > \alpha_c$, and its correct prediction regarding the irrelevance of additive decision noise at the level of macroscopic order parameters for $\alpha > \alpha_c$. Moreover, granted the deterministic approximation that was its starting point and granted the standard tricks and assumptions of the replica method, the subsequent calculation was elegant and formulated in terms of physically sensible and natural quantities.

A limitation of the present replica calculation is that it can only make safe predictions above the critical point, for $\alpha > \alpha_c$. At α_c the susceptibility diverges, and the replica solution brakes down. We have only been able to draw graphs in the regime $\alpha < \alpha_c$ by ad hoc continuations of relations derived for $\alpha > \alpha_c$, the status of which therefore does not extend much beyond wishful thinking, and as a result the predictive quality of these graphs was (not surprisingly) rather poor. Yet, in principle there is no mathematical problem in having $\chi = \infty$ (since, after all, χ was not one of the fundamental original order parameters), provided we retrace our steps to the specific stage in our calculation before the finite χ assumption was introduced and used. This would bring us back to equations (3.41)–(3.45), which continue to hold, but in which for $\alpha < \alpha_c$ we must now find an alternative scaling with β of our order parameters as $\beta \to \infty$. Such an exercise, although leading in a mathematically cleaner way to solutions with $E = 0$, would still be of limited merit, since it follows already from equation (3.63) that, contrary to the observations in our simulation experiments, we can never find volatilities exceeding $\sigma^2 = \frac{1}{2}$. The reason for this is that the high volatilities are caused by oscillations, which no pseudo-equilibrium theory can ever reproduce. A further limitation of the pseudo-equilibrium replica method is that it is unlikely ever to apply to the MG with real as opposed to fake market information, in view of the serious general difficulty to find Lyapunov functions (on which the method ultimately relies) for strongly non-Markovian processes.

Had we not known that the very basis of our calculation was in principle flawed (the assumption of vanishing fluctuations), we would on balance have been rather pleased with our progress at this stage, and probably confident that with further work we would find some way of deriving also the correct theoretical predictions in the non-ergodic (i.e. small α) regime regarding the high volatility solutions and the effects of decision noise. As it is, however, the present replica calculations have in a way only deepened the mystery. Since we are painfully aware that the fluctuations in the MG do not vanish in the limit $N \to \infty$, we are now presented with the additional problem of how to explain *why* this calculation worked so well, at least above the transition point, since there appears to be no obvious reason why it should. It seems intuitively clear that the reason why for $\alpha > \alpha_c$ we got away with neglecting all fluctuations in our present replica analysis must be related to the crucial role played in the MG by the 'frozen agents'. This is borne out by the observations that the phase transition could be defined purely in dimensional terms (i.e. in terms of the fraction of frozen agents relative to α), and that additive decision noise had no impact at all on our macroscopic order parameter equations for $\alpha > \alpha_c$. Frozen agents have expectation values of their microscopic variables $q_i(t)$, which diverge linearly with time; adding finite fluctuations to these diverging expectation values (whether caused by decision noise or otherwise) should indeed asymptotically be of no consequence. This situation is different, however, for those agents who are not frozen, so why then do these 'fickle' agents fail to play a role in macroscopic phenomenology?

We have so far largely followed the historical route towards the pseudo-equilibrium replica calculation. One could in principle try to construct repairs and adaptations aimed at either obtaining the previous results in a more acceptable manner, or at building in the neglected fluctuations. One such adaptation involves studying the stationary state equations for the averages $\langle q_i(t)\rangle$ themselves, leading to exact deterministic microscopic equations in terms of the variables $\{\langle q_i\rangle, \langle\sigma[q_i]\rangle\}$, where we write $\langle f(\mathbf{q})\rangle = \int \mathrm{d}\mathbf{q}\ p_t(\mathbf{q})f(\mathbf{q})$. To see this one may return to the Fokker–Planck equation (3.1) and use it to derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle q_i\rangle &= -\frac{\boldsymbol{\xi}_i\cdot\boldsymbol{\Omega}}{\sqrt{N}} + \frac{1}{N}\sum_j(\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j)\langle\sigma[q_j]\rangle \\ &= -\alpha\frac{\partial}{\partial\langle\sigma[q_i]\rangle}H(\langle\boldsymbol{\sigma}[\mathbf{q}]\rangle) \qquad (3.84)\end{aligned}$$

with again the function $H(\boldsymbol{\sigma})$ as defined in equation (3.10). It is now possible to show that the Lagrange equations corresponding to constrained minimization of $H(\langle\boldsymbol{\sigma}[\mathbf{q}]\rangle)$ are equivalent to the long-time limit of equation (3.84), which allows one to repeat most of the previous replica analysis, and enables the calculation of those stationary state observables that depend only on the quantities $\langle\sigma[q_i]\rangle$, without assuming vanishing fluctuations. In order to describe also the effects of the fluctuations on the volatility, a different approximation was introduced to simplify the diffusion term in the Fokker–Planck equation (3.1) in the stationary state

$$\int \mathrm{d}\mathbf{z}\ P(\mathbf{z})\{A^\mu[\mathbf{q},\mathbf{z}]\}^2 \ \to\ \frac{1}{p}\sum_\nu\left\langle\int \mathrm{d}\mathbf{z}\ P(\mathbf{z})\{A^\nu[\mathbf{q},\mathbf{z}]\}^2\right\rangle = \sigma^2. \qquad (3.85)$$

This implies neglecting in equation (3.1) all dependences of the fluctuations on time and on the variables $\mathbf{q}$ themselves, and reduces the Fokker–Planck equation to

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(\mathbf{q}) = \sum_i\frac{\partial}{\partial q_i}\Big\{p_t(\mathbf{q})\Big[\frac{\boldsymbol{\xi}_i\cdot\boldsymbol{\Omega}}{\sqrt{N}} + \frac{1}{N}\sum_j(\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j)\sigma[q_j]\Big]\Big\}$$

$$+\frac{1}{2}\tilde{\eta}\sigma^2\sum_{ij}\left[\frac{1}{N}(\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j)\right]\frac{\partial^2}{\partial q_i\partial q_j}p_t(\mathbf{q}) \tag{3.86}$$

This equation was subsequently used to derive an approximate equation for the volatility σ. As a result one can now also find solutions of the replica equations in the low α regime which exhibit $\lim_{\alpha\to 0}\sigma > 0$, upon adding a further symmetry breaking term to the Hamiltonian $H(\boldsymbol{\sigma})$. However, this solution is still subject to the bound $\sigma^2 \leq \frac{1}{2}$, so it is not the high volatility state observed in simulations following 'tabula rasa' initialization. One can even progress beyond the RS ansatz (3.35) and calculate non-ergodic stationary states where RS is broken.[12] All this and more has indeed been tried, but in the end it will be clear that such strategies cannot be the most efficient and natural way ahead. The pseudo-equilibrium replica calculation gave us important information on which to build, but has by now brought us as far as can be expected. We need a truly dynamical method, to take into account all relevant fluctuations in all observables of interest (including those which are not functions of the $\{\langle\sigma[q_i]\rangle\}$), to probe the MG in the region $\alpha < \alpha_c$, and to ultimately also analyze the non-Markovian MG versions with real market history.

[12] Replica symmetry breaking (RSB) in itself, i.e. without also incorporating the microscopic fluctuations, will not help us much. One can already deduce this from the simple fact that, for additive decision noise, the noise strength parameter T (which we know from simulation experiments to have an important effect for $\alpha < \alpha_c$) vanished from our present formalism already at stages (3.10) and (3.11), before replicas had even entered the stage.

4. Dynamics of the batch MG with fake memory

One of the main tools for analyzing the dynamics of disordered stochastic many-particle systems is generating functional analysis. This formalism allows one to carry out the disorder average (which in MGs is an average over all strategies) of dynamic macroscopic observables and take the $N \to \infty$ limit exactly.

Here we apply this technique to the MG with fake market information. As a first stage we redefine the original MG equations and choose a so-called 'batch version' of the strategy valuation updates. This will allow us to explain the method in its simplest possible setting, without as yet having to use path integrals. We will do this in considerable detail, allowing us to be more brief on technical details in subsequent chapters. In the original MG with fake market memory there were two sources of fluctuations: those due to the randomly drawn external information, viz. the random variables $\mu(\ell)$ in equation (2.22), and those induced by decision noise, viz. the Gaussian vectors $\mathbf{z}(\ell)$ in equation (2.22). In the batch MG we no longer have the former source of fluctuations, but we retain the latter.

Dynamical studies of disordered systems based on generating functional analysis always consist of two distinct parts. The first part is the derivation in the limit $N \to \infty$ of a set of exact closed equations for dynamic order parameters, which can be interpreted as describing the dynamics of a single 'effective agent'. Due to the disorder in the process, this single-agent acquires an effective 'memory', i.e. will evolve according to a non-trivial non-Markovian stochastic process, even if the original microscopic MG equations are themselves Markovian. The second part is the study of this effective single-agent process.

4.1 Definitions

4.1.1 The batch version of the game

Let us first return to our original (stochastic) laws (2.22) and (2.23) for the MG with two strategies per agent and 'fake' market memory

$$q_i(\ell+1) = q_i(\ell) - \frac{\tilde{\eta}}{\sqrt{N}}\xi_i^{\mu(\ell)}\Big\{\Omega_{\mu(\ell)} + \frac{1}{\sqrt{N}}\sum_j \xi_j^{\mu(\ell)}\sigma[q_j(\ell), z_j(\ell)]\Big\}. \tag{4.1}$$

In Chapter 3 these laws were approximated, in a nutshell, by replacing the microscopic variables $\{q_i\}$ in equation (4.1) by their averages over the random variables $\mu(\ell) \in \{1, \ldots, p\}$ and over the Gaussian random variables $\mathbf{z}(\ell) \in \mathbb{R}^N$ (where $p = \alpha N$). In order to concentrate on the core of the problem, viz. solving the dynamics and dealing with the disorder in a mathematically clean way, we will here replace equation (4.1) by a set of slightly different stochastic equations, which imply that the agents in the MG update their valuations on the basis of the *average* present performance of their strategies over all possible choices of the external information:[13]

$$q_i(t+1) = q_i(t) + \theta_i(t) - \frac{2}{\sqrt{N}}\sum_{\mu=1}^{p}\xi_i^{\mu}\Big\{\Omega_\mu + \frac{1}{\sqrt{N}}\sum_j \xi_j^{\mu}\sigma[q_j(t), z_j(t)]\Big\} \tag{4.2}$$

with $t \in \{0, 1, 2, 3, \ldots\}$. We have also added external perturbation forces $\{\theta_i(t)\}$ in the sense of (2.19). All other definitions, such as that of the decision noise function $\sigma[.,.]$ (2.12) and (2.13) or of the strategy statistics, i.e. the definitions of $\{\xi_i^{\mu}\}$ and $\{\Omega_\mu\}$ (2.10) remain unaltered. We also retain expression (2.23) for the global market bid that would be found upon presenting the fake market memory variable μ, given the actual strategy valuation differences $\mathbf{q}$ and given the decision noise realization $\mathbf{z}$:

$$A^{\mu}[\mathbf{q}, \mathbf{z}] = \Omega_\mu + \frac{1}{\sqrt{N}}\sum_i \sigma[q_i, z_i]\xi_i^{\mu}. \tag{4.3}$$

Thus, apart from the perturbations $\{\theta_i(t)\}$, the only differences between equation (4.2) and the starting point (3.2) of the replica calculation is that in the latter we had continuous time and replaced the variables $\{q_i\}$ by their averages over the decision noise variables. We will in this chapter write averaging over the discrete-time stochastic process (4.2) simply as $\langle \cdots \rangle$, and averages over the Gaussian decision noise variables $\{\mathbf{z}(\ell)\}$ as $\langle \cdots \rangle_{\mathbf{z}}$.

The new definition of our microscopic process (4.2) implies also a redefinition of the volatility σ. Since now by definition *all* p possible values of μ will occur precisely once at each iteration step, the volatility is now to be written as

$$\sigma^2 = \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\tau}\left\{\frac{1}{p}\sum_{\mu=1}^{p}\langle (A^{\mu}[\mathbf{q}(t), \mathbf{z}(t)])^2\rangle - \left(\frac{1}{p}\sum_{\mu=1}^{p}\langle A^{\mu}[\mathbf{q}(t), \mathbf{z}(t)]\rangle\right)^2\right\}. \tag{4.4}$$

[13] In addition we choose $\tilde{\eta} = 2$, which is of no consequence beyond setting a specific scale for the time axis, but will be found to give the simplest equations later.

We will find in our analysis that it will be advantageous to define also a more general object to characterize the global bid fluctuations in the system, namely the volatility matrix $\Xi_{tt'}$:

$$\Xi_{tt'} = \frac{1}{p}\sum_{\mu=1}^{p}\left\langle \left[A^{\mu}[\mathbf{q}(t),\mathbf{z}(t)] - \frac{1}{p}\sum_{\nu=1}^{p}\langle A^{\nu}[\mathbf{q}(t),\mathbf{z}(t)]\rangle\right] \right.$$
$$\left. \times \left[A^{\mu}[\mathbf{q}(t'),\mathbf{z}(t')] - \frac{1}{p}\sum_{\nu=1}^{p}\langle A^{\nu}[\mathbf{q}(t'),\mathbf{z}(t')]\rangle\right]\right\rangle. \tag{4.5}$$

This latter object measures not only the magnitude but also the temporal correlations of the market fluctuations, with $\sigma^2 = \lim_{\tau\to\infty}\tau^{-1}\sum_{t=1}^{\tau}\Xi_{tt}$. In the limit $T \to \infty$ (i.e. purely random decision making), where $\langle\sigma[q_i(t), z_i(t)]\rangle = 0$, we simply find upon substitution of equation (4.3) and upon using the statistical properties of the strategy variables $\{\xi_i^{\mu}, \omega_i^{\mu}\}$ that for $N \to \infty$

$$\begin{aligned}\lim_{N\to\infty}\lim_{T\to\infty}\Xi_{tt'} &= \frac{1}{pN}\sum_{\mu}\sum_{ij}\left\langle\left[\omega_i^{\mu} + \xi_i^{\mu}\sigma[q_i(t), z_i(t)]\right]\left[\omega_j^{\mu} + \xi_j^{\mu}\sigma[q_j(t'), z_j(t')]\right]\right\rangle \\ &= \frac{1}{pN}\sum_{\mu}\sum_{ij}\left[\omega_i^{\mu}\omega_j^{\mu} + \xi_i^{\mu}\xi_j^{\mu}\delta_{ij}\delta_{tt'}\right] \\ &= \frac{1}{2} + \frac{1}{2}\delta_{tt'}.\end{aligned} \tag{4.6}$$

In particular, $\lim_{N\to\infty}\lim_{T\to\infty}\sigma = 1$, as expected.

4.1.2 The generating functional

In contrast to equation (3.2), the present process (4.2) is truly stochastic, so the natural language for our analysis is that of probability densities, similar to equations (2.24) and (2.25). Here the details of the microscopic transition probability density operator $W(\mathbf{q}|\mathbf{q}')$ will be slightly different, however, in that the sum over μ now occurs inside rather than outside the δ-functions and that the operator carries an explicit time dependence. If we write the δ-distributions in the expression for the present kernel $W_t(\mathbf{q}|\mathbf{q}')$ in integral representation, we obtain

$$p_{t+1}(\mathbf{q}) = \int \mathrm{d}\mathbf{q}'\, W_t(\mathbf{q}|\mathbf{q}')p_t(\mathbf{q}'), \tag{4.7}$$

$$W_t(\mathbf{q}|\mathbf{q}') = \int \mathrm{d}\mathbf{z}\, P(\mathbf{z})\prod_i \delta\left[q_i - q_i' - \theta_i(t) + \frac{2}{\sqrt{N}}\sum_{\mu=1}^{p}\xi_i^{\mu}A^{\mu}[\mathbf{q}',\mathbf{z}]\right]$$

$$= \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) \int \frac{\mathrm{d}\hat{\mathbf{q}}}{(2\pi)^N} \mathrm{e}^{\mathrm{i}\sum_i \hat{q}_i \left[q_i - q_i' - \theta_i(t) + \frac{2}{\sqrt{N}} \sum_\mu \xi_i^\mu A^\mu[\mathbf{q}', \mathbf{z}]\right]}, \tag{4.8}$$

with $P(\mathbf{z}) = \prod_i P(z_i)$. The disorder-averaged expectation value at time t of any observable $f(\mathbf{q})$ (which can and often will itself depend explicitly on the disorder) would be written as $\overline{\langle f(\mathbf{q}(t))\rangle} = \int \mathrm{d}\mathbf{q}\, \overline{p_t(\mathbf{q}) f(\mathbf{q})}$.

The moment generating functional for a stochastic process of the type (4.7) (4.8) is a direct generalization to the case of multiple times and multiple components of the more conventional moment generating (or 'characteristic') function $Z[\psi] = \langle \exp[\mathrm{i}\psi x]\rangle$ of a single ordinary random variable x. For a stochastic process with dynamical variables $\mathbf{q}(t)$ this generating functional would normally be defined as $Z[\boldsymbol{\psi}] = \langle \exp[\mathrm{i}\sum_{t\geq 0}\sum_i \psi_i(t) q_i(t)]\rangle$. In the MG, however, the main quantities of interest ultimately depend on the actual bids submitted by our agents, i.e. on the variables $\sigma[q_i(t), z_i(t)]$, so instead we will define

$$Z[\boldsymbol{\psi}] = \left\langle \mathrm{e}^{\mathrm{i}\sum_{t\geq 0}\sum_i \psi_i(t)\sigma[q_i(t), z_i(t)]}\right\rangle. \tag{4.9}$$

Here $\boldsymbol{\psi}$ is taken to denote the full collection of real-valued variables $\{\psi_i(t)\}$, at all sites i and all times t. The brackets denote an average over the stochastic process (4.7). Since equation (4.9) involves multiple times, this average is in effect an average over all possible 'paths' of the microscopic state vector $\mathbf{q}$, where each path $\{\mathbf{q}(0), \mathbf{q}(1), \mathbf{q}(2), \ldots\}$ has probability density $p(\mathbf{q}(0), \mathbf{q}(1), \mathbf{q}(2), \ldots) = p_0(\mathbf{q}(0)) \prod_{t\geq 0} W_t(\mathbf{q}(t{+}1)|\mathbf{q}(t))$.[14] Note that for equation (4.9) to be well defined we need to specify an upper limit $t_{\max}$ in the summation over times; unless indicated otherwise, all time summations and time products will therefore henceforth run from $t = 0$ to $t = t_{\max}$. By taking suitable derivatives of the generating functional (4.9) with respect to the variables $\{\psi_i(t)\}$ one can generate all moments of the random variables $\{\sigma[q_i(t), z_i(t)]\}$, at arbitrary times, including their derivatives with respect to perturbations. For instance

$$\langle \sigma[q_i(t), z_i(t)]\rangle = -\mathrm{i} \lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial}{\partial\psi_i(t)} Z[\boldsymbol{\psi}], \tag{4.10}$$

$$\langle \sigma[q_i(t), z_i(t)]\sigma[q_j(t'), z_j(t')]\rangle = -\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial^2}{\partial\psi_i(t)\partial\psi_j(t')} Z[\boldsymbol{\psi}], \tag{4.11}$$

$$\frac{\partial}{\partial\theta_j(t')}\langle \sigma[q_i(t), z_i(t)]\rangle = -\mathrm{i} \lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial^2}{\partial\psi_i(t)\partial\theta_j(t')} Z[\boldsymbol{\psi}]. \tag{4.12}$$

Alternatively, one could also view equation (4.9) simply as the Fourier transform of the joint probability density $p(\mathrm{s}(0), \mathrm{s}(1), \mathrm{s}(2), \ldots)$ for finding the N-component vector

[14] We can now appreciate the mathematical advantage of the present 'batch' dynamics, with discrete time, over the continuous time 'on-line' alternative: in the latter case the integrals over paths $\{\mathbf{q}(0), \mathbf{q}(1), \mathbf{q}(2), \ldots\}$ would be replaced by path integrals over continuous trajectories $\{\mathbf{q}(t)\}$ with $t \in \mathbb{R}$.

$\mathbf{s} = (\sigma[q_1, z_1], \ldots, \sigma[q_N, z_N])$ taking any given path $\{\mathbf{s}(0), \mathbf{s}(1), \mathbf{s}(2), \ldots\}$ through state space, from which this joint probability density can in principle be recovered via an inverse Fourier transformation

$$\int \frac{\mathrm{d}\boldsymbol{\psi}(0) \cdots \mathrm{d}\boldsymbol{\psi}(t_{\max})}{(2\pi)^{t_{\max}+1}} Z[\boldsymbol{\psi}] \mathrm{e}^{-\mathrm{i}\sum_t \psi_i(t) s_i(t)} = \left\langle \prod_i \prod_t \delta[s_i(t) - \sigma[q_i(t), z_i(t)]] \right\rangle = p(\mathbf{s}(0), \ldots, \mathbf{s}(t_{\max})). \tag{4.13}$$

Either way, we may safely conclude that $Z[\boldsymbol{\psi}]$ contains all the information we are interested in. Averaging (4.9) over the disorder (i.e. over the microscopic realization of the strategies) allows us to obtain from equations (4.10–4.12) similar statements in terms of disorder-averaged observables. In particular, we will find a prominent role in our dynamical theory for the asymptotic disorder-averaged single-site correlation and response functions $C_{tt'}$ and $G_{tt'}$, which are defined as

$$C_{tt'} = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \sigma[q_i(t), z_i(t)] \sigma[q_i(t'), z_i(t')] \rangle} = -\lim_{N\to\infty} \lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N} \sum_i \frac{\partial^2}{\partial\psi_i(t)\partial\psi_i(t')} \overline{Z[\boldsymbol{\psi}]}, \tag{4.14}$$

$$G_{tt'} = \lim_{N\to\infty} \frac{1}{N} \sum_i \frac{\partial}{\partial\theta_i(t')} \overline{\langle \sigma[q_i(t), z_i(t)] \rangle} = -\mathrm{i} \lim_{N\to\infty} \lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N} \sum_i \frac{\partial^2}{\partial\psi_i(t)\partial\theta_i(t')} \overline{Z[\boldsymbol{\psi}]}. \tag{4.15}$$

4.1.3 Generating functional analysis and disorder

The property that makes the generating functional (4.9) an ideal tool with which to achieve our objective of solving the dynamics of the MG is that its disorder average $\overline{Z[\boldsymbol{\psi}]}$ can be calculated relatively easily. All that is required is for us to define suitable auxiliary quantities (or fields), via δ-distributions, whose purpose is to transport the disorder variables into exponentials. The simple relation $\overline{Z[\mathbf{0}]} = 1$ ensures that, in contrast to the situation with the measure (3.24), there will be no awkward normalization factor that contains the disorder. This removes the need for replica theory.[15]

[15] The generating functional method was in fact initially presented in literature as a mathematically more respectable and acceptable alternative to the replica method.

Let us now be more explicit and show precisely how this preparatory work is done in practice. We insert first equation (4.8) into equation (4.9) and subsequently substitute the definition (4.3), with the shorthand $s_i(t) = \sigma[q_i(t), z_i(t)]$:

$$Z[\boldsymbol{\psi}] = \left\langle \int p_0(\mathbf{q}(0)) \Big[\prod_{it} \frac{\mathrm{d}q_i(t)\mathrm{d}\hat{q}_i(t)}{2\pi} \mathrm{e}^{\mathrm{i}\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)]+\mathrm{i}\psi_i(t)s_i(t)} \Big] \right.$$
$$\left. \times \prod_{i\mu} \mathrm{e}^{\frac{2i}{\sqrt{N}}\xi_i^\mu \sum_t \hat{q}_i(t) A^\mu[\mathbf{q}(t),\mathbf{z}(t)]} \right\rangle_{\mathbf{z}}$$
$$= \left\langle \int p_0(\mathbf{q}(0)) \Big[\prod_{it} \frac{\mathrm{d}q_i(t)\mathrm{d}\hat{q}_i(t)}{2\pi} \mathrm{e}^{\mathrm{i}\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)]+\mathrm{i}\psi_i(t)s_i(t)} \Big] \right.$$
$$\left. \times \prod_{ij\mu} \mathrm{e}^{\frac{2i}{N}\xi_i^\mu \sum_t \hat{q}_i(t)[\omega_j^\mu + s_j(t)\xi_j^\mu]} \right\rangle_{\mathbf{z}} \qquad (4.16)$$

In order to decouple all occurrences of the disorder variables $\{\xi_i^\mu, \omega_i^\mu\}$ we introduce the two shorthands $w_t^\mu = (2/N)^{\frac{1}{2}} \sum_i \hat{q}_i(t)\xi_i^\mu$ and $x_t^\mu = (2/N)^{\frac{1}{2}} \sum_i s_i(t)\xi_i^\mu$, via substitution into the generating functional of integrals over appropriate δ-functions:

$$1 = \prod_{\mu t} \int \mathrm{d}w_t^\mu \mathrm{d}x_t^\mu \; \delta\Big[w_t^\mu - \frac{\sqrt{2}}{\sqrt{N}} \sum_i \hat{q}_i(t)\xi_i^\mu\Big] \delta\Big[x_t^\mu - \frac{\sqrt{2}}{\sqrt{N}} \sum_i s_i(t)\xi_i^\mu\Big].$$

Writing these δ-functions in their integral representations, via $\delta(u) = \int [\mathrm{d}\hat{u}/2\pi]\mathrm{e}^{\mathrm{i}\hat{u}u}$, subsequently generates conjugate integration variables $\{\hat{w}_t^\mu, \hat{x}_t^\mu\}$. Upon also abbreviating $\mathcal{D}\mathbf{q} = \prod_{it}[\mathrm{d}q_i(t)/\sqrt{2\pi}]$, $\mathcal{D}\mathbf{w} = \prod_{\mu t}[\mathrm{d}w_t^\mu/\sqrt{2\pi}]$, and $\mathcal{D}\mathbf{x} = \prod_{\mu t}[\mathrm{d}x_t^\mu/\sqrt{2\pi}]$ (with similar definitions for $\mathcal{D}\hat{\mathbf{q}}$, $\mathcal{D}\hat{\mathbf{w}}$ and $\mathcal{D}\hat{\mathbf{x}}$, respectively), the generating functional is seen to take the following form

$$Z[\boldsymbol{\psi}] = \int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}} \; \mathrm{e}^{\mathrm{i}\sum_{t\mu}[\hat{w}_t^\mu w_t^\mu + \hat{x}_t^\mu x_t^\mu + w_t^\mu(\sqrt{2}\Omega_\mu + x_t^\mu)]}$$
$$\times \int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}} \; p_0(\mathbf{q}(0)) \; \langle \mathrm{e}^{\frac{-\mathrm{i}\sqrt{2}}{\sqrt{N}} \sum_{\mu i} \xi_i^\mu \sum_t [\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)] + \mathrm{i}\sum_{it}\psi_i(t)s_i(t)} \rangle_{\mathbf{z}}$$
$$\times \mathrm{e}^{\mathrm{i}\sum_{ti}\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)]} \qquad (4.17)$$

with, as always, $\Omega_\mu = N^{-\frac{1}{2}} \sum_i \omega_i^\mu$. The disorder variables $\{\xi_i^\mu, \omega_i^\mu\}$, and therefore also the strategy look-up table entries $\{R_\mu^{ia}\}$, are now seen to appear linearly in exponentials only, so averaging $Z[\boldsymbol{\psi}]$ in its present form (4.17) over the disorder has indeed become trivial.

4.2 The disorder-averaged generating functional

4.2.1 Evaluation of the disorder average

We can now calculate the disorder average $\overline{Z[\boldsymbol{\psi}]}$ of equation (4.17), upon restoring the definitions of $\xi_i^\mu = \frac{1}{2}(R_\mu^{i1} - R_\mu^{i2})$ and $\Omega_\mu = \frac{1}{2}N^{-\frac{1}{2}}\sum_j (R_\mu^{j1} + R_\mu^{j2})$. We recall that the $\{R_\mu^{ia}\}$ are all statistically independent, each taking the values ± 1 with equal probabilities. If we restrict ourselves to times $t_{\max}$ that do not scale with N, we may write the relevant disorder-dependent part of $\overline{Z[\boldsymbol{\psi}]}$ as

$$
\begin{aligned}
&\overline{\mathrm{e}^{\mathrm{i}\sqrt{2}\sum_{t\mu} w_t^\mu \Omega_\mu - \frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{\mu i}\xi_i^\mu \sum_t [\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)]}} \\
&\quad = \prod_{i\mu} \overline{\mathrm{e}^{\frac{\mathrm{i}}{\sqrt{2N}}\sum_t \left[(R_\mu^{i1}+R_\mu^{i2})w_t^\mu - (R_\mu^{i1}-R_\mu^{i2})[\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)]\right]}} \\
&\quad = \prod_{i\mu}\left\{\cos\left[\frac{1}{\sqrt{2N}}\sum_t [w_t^\mu - \hat{w}_t^\mu \hat{q}_i(t) - \hat{x}_t^\mu s_i(t)]\right]\right. \\
&\qquad \left.\times \cos\left[\frac{1}{\sqrt{2N}}\sum_t [w_t^\mu + \hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)]\right]\right\} \\
&\quad = \prod_{i\mu} \mathrm{e}^{-\frac{1}{4N}(\sum_t [w_t^\mu - \hat{w}_t^\mu \hat{q}_i(t) - \hat{x}_t^\mu s_i(t)])^2 - \frac{1}{4N}(\sum_t [w_t^\mu + \hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)])^2 + \mathcal{O}(N^0)} \\
&\quad = \prod_{i\mu} \mathrm{e}^{-\frac{1}{2N}\left\{(\sum_t w_t^\mu)^2 + (\sum_t [\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)])^2\right\} + \mathcal{O}(N^0)} \\
&\quad = \mathrm{e}^{-\frac{1}{2}\sum_{\mu t t'}\left[w_t^\mu w_{t'}^\mu + \hat{w}_t^\mu L_{tt'} \hat{w}_{t'}^\mu + 2\hat{x}_t^\mu K_{tt'} \hat{w}_{t'}^\mu + \hat{x}_t^\mu C_{tt'} \hat{x}_{t'}^\mu\right] + \mathcal{O}(N^0)},
\end{aligned}
\tag{4.18}
$$

where we introduced the temporary abbreviations

$$
C_{tt'} = \frac{1}{N}\sum_i s_i(t)s_i(t'), \qquad K_{tt'} = \frac{1}{N}\sum_i s_i(t)\hat{q}_i(t'), \qquad L_{tt'} = \frac{1}{N}\sum_i \hat{q}_i(t)\hat{q}_i(t').
$$

We isolate these two-time functions[16] in the usual manner by inserting appropriate δ-functions (in integral representation, which generates associated conjugate integration variables), and define the corresponding shorthands $\mathcal{D}C = \prod_{tt'}[\mathrm{d}C_{tt'}/\sqrt{2\pi}]$, $\mathcal{D}K = \prod_{tt'}[\mathrm{d}K_{tt'}/\sqrt{2\pi}]$, and $\mathcal{D}L = \prod_{tt'}[\mathrm{d}L_{tt'}/\sqrt{2\pi}]$ (with similar definitions for $\mathcal{D}\hat{C}$, $\mathcal{D}\hat{K}$, and $\mathcal{D}\hat{L}$, respectively). As in the case of the replica calculations in the previous chapter, we must ensure that those new quantities in the exponentials of which the scaling with N is still open will scale similarly to those terms where we have no choice, i.e. they

[16] Note that we chose to use again the notation $C_{tt'}$, which has already been assigned a meaning via equation (4.14). This has been done for reasons of economy, since we will find below that in the limit $N \to \infty$ the two will indeed become identical.

must all be of order N. This implies that we will therefore insert the following terms into equation (4.17):

$$1 = N^{(t_{\max}+1)^2} \int \mathcal{D}C\mathcal{D}\hat{C}\ \mathrm{e}^{\mathrm{i}N\sum_{tt'}\hat{C}_{tt'}[C_{tt'}-\frac{1}{N}\sum_i s_i(t)s_i(t')]}, \tag{4.19}$$

$$1 = N^{(t_{\max}+1)^2} \int \mathcal{D}K\mathcal{D}\hat{K}\ \mathrm{e}^{\mathrm{i}N\sum_{tt'}\hat{K}_{tt'}[K_{tt'}-\frac{1}{N}\sum_i s_i(t)\hat{q}_i(t')]}, \tag{4.20}$$

$$1 = N^{(t_{\max}+1)^2} \int \mathcal{D}L\mathcal{D}\hat{L}\ \mathrm{e}^{\mathrm{i}N\sum_{tt'}\hat{L}_{tt'}[L_{tt'}-\frac{1}{N}\sum_i \hat{q}_i(t)\hat{q}_i(t')]}. \tag{4.21}$$

Note that now the quantities $\{C, K, L\}$ are no longer shorthands, but have become integration variables. Substitution into $\overline{Z[\boldsymbol{\psi}]}$, followed by some simple cosmetic rearranging of terms, then leads us immediately to

$$\begin{aligned}
\overline{Z[\boldsymbol{\psi}]} = & \int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\ \mathrm{e}^{\mathrm{i}N\sum_{tt'}[\hat{C}_{tt'}C_{tt'}+\hat{K}_{tt'}K_{tt'}+\hat{L}_{tt'}L_{tt'}]+\mathcal{O}(\log(N))} \\
& \times \int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\ \mathrm{e}^{\mathrm{i}\sum_{t\mu}[\hat{w}_t^\mu w_t^\mu+\hat{x}_t^\mu x_t^\mu+w_t^\mu x_t^\mu]} \\
& \times \mathrm{e}^{-\frac{1}{2}\sum_{\mu tt'}\left[w_t^\mu w_{t'}^\mu+\hat{w}_t^\mu L_{tt'}\hat{w}_{t'}^\mu+2\hat{x}_t^\mu K_{tt'}\hat{w}_{t'}^\mu+\hat{x}_t^\mu C_{tt'}\hat{x}_{t'}^\mu\right]} \\
& \times \int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}}\ p_0(\mathbf{q}(0))\ \langle \mathrm{e}^{\mathrm{i}\sum_{ti}\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)]+\mathrm{i}\sum_{it}\psi_i(t)s_i(t)} \\
& \times \mathrm{e}^{-\mathrm{i}\sum_i\sum_{tt'}[\hat{C}_{tt'}s_i(t)s_i(t')+\hat{K}_{tt'}s_i(t)\hat{q}_i(t')+\hat{L}_{tt'}\hat{q}_i(t)\hat{q}_i(t')]}\rangle_{\mathbf{z}}.
\end{aligned} \tag{4.22}$$

We see that upon assuming simple initial conditions of the form $p_0(\mathbf{q}) = \prod_i p_0(q_i)$, the i-dependent terms in the disorder-averaged generating functional (4.22) will factorize fully over the agents i, and we arrive at

$$\overline{Z[\boldsymbol{\psi}]} = \int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\ \mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(\log(N))} \tag{4.23}$$

with, in view of $\mu = 1, \ldots, \alpha N$,

$$\Psi = \mathrm{i}\sum_{tt'}[\hat{C}_{tt'}C_{tt'} + \hat{K}_{tt'}K_{tt'} + \hat{L}_{tt'}L_{tt'}], \tag{4.24}$$

$$\begin{aligned}
\Phi = & \alpha \log\Bigg[\int \mathcal{D}w\mathcal{D}\hat{w}\mathcal{D}x\mathcal{D}\hat{x}\ \mathrm{e}^{\mathrm{i}\sum_t[\hat{w}_t w_t+\hat{x}_t x_t+w_t x_t]} \\
& \times \mathrm{e}^{-\frac{1}{2}\sum_{tt'}[w_t w_{t'}+\hat{w}_t L_{tt'}\hat{w}_{t'}+2\hat{x}_t K_{tt'}\hat{w}_{t'}+\hat{x}_t C_{tt'}\hat{x}_{t'}]}\Bigg],
\end{aligned} \tag{4.25}$$

$$\Omega = \frac{1}{N}\sum_i \log\Big\langle \int \mathcal{D}q\mathcal{D}\hat{q}\ p_0(q(0))$$

$$\times\, \mathrm{e}^{\mathrm{i}\sum_t[\hat{q}(t)[q(t+1)-q(t)-\theta_i(t)]+\psi_i(t)\sigma[q(t),z(t)]]-\mathrm{i}\sum_{tt'}\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')}$$
$$\times\, \mathrm{e}^{-\mathrm{i}\sum_{tt'}[\hat{C}_{tt'}\sigma[q(t),z(t)]\sigma[q(t'),z(t')]+\hat{K}_{tt'}\sigma[q(t),z(t)]\hat{q}(t')]}\Big\rangle_{\mathbf{z}}. \tag{4.26}$$

The sub-dominant $\mathcal{O}(\log(N))$ term in the exponent of equation (4.23) is generated by the various factors of N in front of the integrals (4.19)–(4.21) and by the sub-dominant contribution to the exponent of (4.18), and is therefore independent of the generating fields $\{\psi_i(t), \theta_i(t)\}$. The contribution (4.24) is a 'bookkeeping' term, linking the dynamic order parameters to their conjugates. The term (4.25) reflects the statistical properties of the agents' strategies, and boils down to a Gaussian integral. Finally, in the non-trivial term (4.26) we see that the decision noise averages have been reduced to single-site (but multiple-time) ones: $\langle g[z(1), z(2), \ldots]\rangle_{\mathbf{z}} = \int\prod_t[\mathrm{d}z(t)P(z(t))]\; g[z(1), z(2), \ldots]$.

4.2.2 The saddle-point equations

Since all the quantities appearing in equation (4.23) are seen to be macroscopic (the individual microscopic variables have been integrated out) and since all three functions $\{\Psi, \Phi, \Omega\}$ are seen to scale with the system size as $\mathcal{O}(N^0)$, we reach the important conclusion that equation (4.23) can be evaluated by steepest descent integration, leading to coupled saddle-point equations from which to solve the dynamic order parameters kernels $\{C, \hat{C}, K, \hat{K}, L, \hat{L}\}$. As in our previous replica calculations, we will here again encounter imaginary saddle-points. As before, these are dealt with by shifting integration contours in the space of $\{C, \hat{C}, K, \hat{K}, L, \hat{L}\}$ (see our previous notes on the mathematical background to this). Extremization of the extensive exponent $N[\Psi + \Phi + \Omega]$ of equation (4.23) with respect to $\{C, \hat{C}, K, \hat{K}, L, \hat{L}\}$ is straightforward and gives the following saddle-point equations

$$C_{tt'} = \langle\, \sigma[q(t), z(t)]\; \sigma[q(t'), z(t')]\, \rangle_\star, \tag{4.27}$$

$$K_{tt'} = \langle\, \sigma[q(t), z(t)]\; \hat{q}(t')\, \rangle_\star, \tag{4.28}$$

$$L_{tt'} = \langle\, \hat{q}(t)\hat{q}(t')\, \rangle_\star, \tag{4.29}$$

$$\hat{C}_{tt'} = \frac{i\partial\Phi}{\partial C_{tt'}}, \quad \hat{K}_{tt'} = \frac{i\partial\Phi}{\partial K_{tt'}}, \quad \hat{L}_{tt'} = \frac{i\partial\Phi}{\partial L_{tt'}}. \tag{4.30}$$

The notation $\langle\cdots\rangle_\star$ in the above expressions is at this stage just a shorthand for the following operation (since it still contains imaginary terms, we are not yet allowed to interpret it as an average or expectation value)

$$\langle g[\{q, \hat{q}, z\}]\rangle_\star = \lim_{N\to\infty}\frac{1}{N}\sum_i \langle g[\{q, \hat{q}, z\}]\rangle_i, \tag{4.31}$$

$$\langle g[\{q,\hat{q},z\}]\rangle_{i}, = \frac{\int \mathcal{D}q\mathcal{D}\hat{q}\ \langle M_i[\{q,\hat{q},z\}]g[\{q,\hat{q},z\}]\rangle_{\mathbf{z}}}{\int \mathcal{D}q\mathcal{D}\hat{q}\ \langle M_i[\{q,\hat{q},z\}]\rangle_{\mathbf{z}}}, \tag{4.32}$$

$$\begin{aligned} M_i[\{q,\hat{q},z\}] &= p_0(q(0))\ \mathrm{e}^{\mathrm{i}\sum_t \hat{q}(t)[q(t+1)-q(t)-\theta_i(t)]+\mathrm{i}\sum_t \psi_i(t)\sigma[q(t),z(t)]} \\ &\quad\times \mathrm{e}^{-\mathrm{i}\sum_{tt'}[\hat{C}_{tt'}\sigma[q(t),z(t)]\sigma[q(t'),z(t')]+\hat{K}_{tt'}\sigma[q(t),z(t)]\hat{q}(t')]} \\ &\quad\times \mathrm{e}^{-\mathrm{i}\sum_{tt'}\hat{L}_{tt'}\hat{q}(t)\hat{q}(t')}. \end{aligned} \tag{4.33}$$

4.3 Simplification of the saddle-point equations

Our saddle-point equations (4.27)–(4.30) seem somewhat unwieldy, but they can be simplified in a number of ways. Firstly, after deriving the physical interpretation of our dynamic order parameters we can discard the generating fields $\{\psi_i(t),\theta_i(t)\}$. Secondly, we will find that a number of the order parameters must in fact be zero. Thirdly, we can carry out the integrals in Φ (4.25) and subsequently work out equations (4.30) explicitly. Fourth, we will show how the conjugate variables $\{\hat{q}\}$, as appearing in equation (4.28) and (4.29), can be transformed into derivatives with respect to the perturbation fields, as a result of which the effective measure (4.33) can be simplified considerably.

4.3.1 Identification of order parameters

Let us return to our two expressions (4.14) and (4.15) for the correlation and response functions C and G, and insert into the latter our formula (4.23) for the disorder-averaged generating functional. Let us also insert equation (4.23) into the trivial identity $\partial^2\overline{Z}[\mathbf{0}]/\partial\theta_i(t)\partial\theta_i(t')=0$. In doing so we will make use of the fact that only the term Ω in the exponent of equation (4.23) depends on the fields $\{\psi_i,\theta_i\}$. In working out the resulting three equations we encounter the following derivatives

$$\begin{aligned} \frac{\partial\Omega}{\partial\psi_i(t)} &= \frac{i}{N}\langle\sigma[q(t),z(t)]\rangle_i \\ \frac{\partial^2\Omega}{\partial\psi_i(t)\partial\psi_i(t')} &= -\frac{1}{N}\Big\{\langle\sigma[q(t),z(t)]\sigma[q(t'),z(t')]\rangle_i \\ &\qquad -\langle\sigma[q(t),z(t)]\rangle_i\langle\sigma[q(t'),z(t')]\rangle_i\Big\} \\ \frac{\partial^2\Omega}{\partial\psi_i(t)\partial\theta_i(t')} &= \frac{1}{N}\Big\{\langle\sigma[q(t),z(t)]\hat{q}(t')\rangle_i-\langle\sigma[q(t),z(t)]\rangle_i\langle\hat{q}(t')\rangle_i\Big\} \\ \frac{\partial^2\Omega}{\partial\theta_i(t)\partial\theta_i(t')} &= -\frac{1}{N}\Big\{\langle\hat{q}(t)\hat{q}(t')\rangle_i-\langle\hat{q}(t)\rangle_i\langle\hat{q}(t')\rangle_i\Big\}. \end{aligned}$$

Using the shorthand $\{\mathcal{D}\} = \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}K\mathcal{D}\hat{K}\mathcal{D}L\mathcal{D}\hat{L}$ and the normalization identity $\lim_{\boldsymbol{\psi}\to\mathbf{0}} \overline{Z[\boldsymbol{\psi}]} = \langle 1\rangle = 1$ we now find

$$
\begin{aligned}
C_{tt'} &= -\lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N}\sum_i \frac{\int\{\mathcal{D}\}\frac{\partial^2}{\partial\psi_i(t)\partial\psi_i(t')}\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\,\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= -\lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N}\sum_i \frac{\int\{\mathcal{D}\}\left[\frac{\partial^2 N\Phi}{\partial\psi_i(t)\partial\psi_i(t')}+\frac{\partial N\Phi}{\partial\psi_i(t)}\frac{\partial N\Phi}{\partial\psi_i(t')}\right]\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\,\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= \lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\int\{\mathcal{D}\}\frac{1}{N}\sum_i\langle\sigma[q(t),z(t)]\sigma[q(t'),z(t')]\rangle_i\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\,\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= \lim_{\boldsymbol{\psi}\to\mathbf{0}}\langle\sigma[q(t),z(t)]\sigma[q(t'),z(t')]\rangle_\star, \qquad (4.34)
\end{aligned}
$$

$$
\begin{aligned}
G_{tt'} &= -\mathrm{i}\lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N}\sum_i \frac{\int\{\mathcal{D}\}\frac{\partial^2}{\partial\psi_i(t)\partial\theta_i(t')}\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\,\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= -\mathrm{i}\lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N}\sum_i \frac{\int\{\mathcal{D}\}\left[\frac{\partial^2 N\Phi}{\partial\psi_i(t)\partial\theta_i(t')}+\frac{\partial N\Phi}{\partial\psi_i(t)}\frac{\partial N\Phi}{\partial\theta_i(t')}\right]\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\,\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= -\mathrm{i}\lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\int\{\mathcal{D}\}\frac{1}{N}\sum_i\langle\sigma[q(t),z(t)]\hat{q}(t')\rangle_i\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\,\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= -\mathrm{i}\lim_{\boldsymbol{\psi}\to\mathbf{0}}\langle\sigma[q(t),z(t)]\hat{q}(t')\rangle_\star, \qquad (4.35)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N}\sum_i \frac{\int\{\mathcal{D}\}\frac{\partial^2}{\partial\theta_i(t)\partial\theta_i(t')}\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\,\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= \lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N}\sum_i \frac{\int\{\mathcal{D}\}\left[\frac{\partial^2 N\Phi}{\partial\theta_i(t)\partial\theta_i(t')}+\frac{\partial N\Phi}{\partial\theta_i(t)}\frac{\partial N\Phi}{\partial\theta_i(t')}\right]\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\,\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= -\lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\int\{\mathcal{D}\}1/N\sum_i\langle\hat{q}(t)\hat{q}(t')\rangle_i\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\,\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= -\lim_{\boldsymbol{\psi}\to\mathbf{0}}\langle\hat{q}(t)\hat{q}(t')\rangle_\star. \qquad (4.36)
\end{aligned}
$$

The final expressions (4.34)–(4.36) are to be evaluated *at* the physical saddle-point of $\Psi+\Omega+\Phi$. We conclude from equation (4.34) that our previous claim regarding the correlation function interpretation of equation (4.27) was indeed correct. From equations (4.35) and (4.36) we extract, in combination with equations (4.28) and (4.29) that for all (t,t')

$$K_{tt'} = iG_{tt'}, \quad L_{tt'} = 0. \tag{4.37}$$

At this stage we may send the fields $\{\psi_i\}$ to zero, and we choose the perturbations $\{\theta_i\}$ to be independent of i. This results in the measure $M_i[\{q, \hat{q}, z\}]$ losing its dependence on i, and our saddle-point equations (4.27)–(4.30) are consequently simplified to

$$C_{tt'} = \langle\, \sigma[q(t), z(t)]\, \sigma[q(t'), z(t')]\,\rangle_\star, \tag{4.38}$$

$$G_{tt'} = -\mathrm{i}\langle\, \sigma[q(t), z(t)]\, \hat{q}(t')\,\rangle_\star, \tag{4.39}$$

$$\hat{C}_{tt'} = \lim_{L\to 0} \frac{\mathrm{i}\partial\Phi}{\partial C_{tt'}}, \quad \hat{K}_{tt'} = \lim_{L\to 0} \frac{\mathrm{i}\partial\Phi}{\partial K_{tt'}}, \quad \hat{L}_{tt'} = \lim_{L\to 0} \frac{\mathrm{i}\partial\Phi}{\partial L_{tt'}} \tag{4.40}$$

with

$$\langle g[\{q, \hat{q}, z\}]\rangle_\star = \frac{\int \mathcal{D}q\mathcal{D}\hat{q}\, \langle M[\{q, \hat{q}, z\}] g[\{q, \hat{q}, z\}]\rangle_{\mathbf{z}}}{\int \mathcal{D}q\mathcal{D}\hat{q}\, \langle M[\{q, \hat{q}, z\}]\rangle_{\mathbf{z}}}, \tag{4.41}$$

$$\begin{aligned} M[\{q, \hat{q}, z\}] = {} & p_0(q(0))\, \mathrm{e}^{\mathrm{i}\sum_t \hat{q}(t)[q(t+1)-q(t)-\theta(t)]} \\ & \times \mathrm{e}^{-\mathrm{i}\sum_{tt'}[\hat{C}_{tt'}\sigma[q(t),z(t)]\sigma[q(t'),z(t')]+\hat{K}_{tt'}\sigma[q(t),z(t)]\hat{q}(t')]} \\ & \times \mathrm{e}^{-\mathrm{i}\sum_{tt'}\hat{L}_{tt'}\hat{q}(t)\hat{q}(t')}. \end{aligned} \tag{4.42}$$

4.3.2 Evaluation of Φ

Next we turn our attention to the function Φ (4.25), which, according to equation (4.40), we only need to know for small L. Integration over the variables $\{x_t\}$ in equation (4.25) immediately yields $\prod_t[\sqrt{2\pi}\delta[\hat{x}_t + w_t]]$, so that

$$\Phi = \alpha \log \int \mathcal{D}w\mathcal{D}\hat{w}\, \mathrm{e}^{\mathrm{i}\sum_t \hat{w}_t w_t - \frac{1}{2}\sum_{tt'}[w_t w_{t'} + \hat{w}_t L_{tt'}\hat{w}_{t'} - 2iw_t G_{tt'}\hat{w}_{t'} + w_t C_{tt'} w_{t'}]}. \tag{4.43}$$

It will be efficient to define the two matrices $\mathbf{1}$ and D, with entries $\mathbf{1}_{tt'} = \delta_{tt'}$ and $D_{tt'} = 1 + C_{tt'}$, respectively. The integration over $\{w_t\}$ now gives (see appendix B for multi-variate Gaussian integrals), with the notation $G^\dagger_{tt'} = G_{t't}$:

$$\begin{aligned} \Phi = {} & \alpha \log \int \prod_t \left[\frac{\mathrm{d}\hat{w}_t}{\sqrt{2\pi}}\right] \mathrm{e}^{-\frac{1}{2}\sum_{tt'}\hat{w}_t L_{tt'}\hat{w}_{t'} - \frac{1}{2}\sum_{tt'}\hat{w}_t\left[(\mathbf{1}+G)^\dagger D^{-1}(\mathbf{1}+G)\right]_{tt'}\hat{w}_{t'}} \\ & -\frac{1}{2}\alpha \log\det D \\ = {} & \alpha \log \int \prod_t \left[\frac{\mathrm{d}\hat{w}_t}{\sqrt{2\pi}}\right] \frac{1-(1/2)\sum_{tt'}\hat{w}_t L_{tt'}\hat{w}_{t'} + \mathcal{O}(L^2)}{\det^{-1/2}\left[(\mathbf{1}+G)^\dagger D^{-1}(\mathbf{1}+G)\right]} \mathrm{e}^{-\frac{1}{2}\sum_{tt'}\hat{w}_t\left[(\mathbf{1}+G)^\dagger D^{-1}(\mathbf{1}+G)\right]_{tt'}\hat{w}_{t'}} \\ & -\frac{1}{2}\alpha \log\det D - \frac{1}{2}\alpha \log\det[(\mathbf{1}+G)^\dagger D^{-1}(\mathbf{1}+G)] \end{aligned}$$

$$
\begin{aligned}
&= \alpha \log \left\{ 1 - \frac{1}{2} \sum_{tt'} L_{tt'} \int \prod_t \left[\frac{\mathrm{d}\hat{w}_t}{\sqrt{2\pi}} \right] \frac{\hat{w}_t \hat{w}_{t'} \mathrm{e}^{-(1/2)\sum_{tt'} \hat{w}_t [(\mathbf{1}+G)^\dagger D^{-1}(\mathbf{1}+G)]_{tt'} \hat{w}_{t'}}}{\det^{-1/2} [(\mathbf{1}+G)^\dagger D^{-1} (\mathbf{1}+G)]} \right\} \\
&\quad - \frac{1}{2} \alpha \log \det [(\mathbf{1}+G)^\dagger (\mathbf{1}+G)] + \mathcal{O}(L^2) \\
&= -\alpha \mathrm{Tr} \log(\mathbf{1}+G) - \frac{1}{2} \alpha \sum_{tt'} L_{tt'} [(\mathbf{1}+G)^\dagger D^{-1} (\mathbf{1}+G)]^{-1}_{tt'} + \mathcal{O}(L^2),
\end{aligned} \tag{4.44}
$$

where we have also used the general matrix identity $\log \det U = \mathrm{Tr} \log U$. Expression (4.44) allows us to write the saddle-point equation (4.40) in fully explicit form, using the further general matrix identity $\partial \mathrm{Tr} \log U / \partial U_{tt'} = (U^{-1})_{t't}$:

$$\hat{C}_{tt'} = 0, \tag{4.45}$$

$$\hat{K}_{tt'} = -\alpha (\mathbf{1}+G)^{-1}_{t't}, \tag{4.46}$$

$$\hat{L}_{tt'} = -\frac{1}{2} i\alpha [(\mathbf{1}+G)^{-1} D (\mathbf{1}+G^\dagger)^{-1}]_{tt'}. \tag{4.47}$$

These results lead to a further simplification and compactification of our saddle-point equations (4.38)–(4.40), which now involve only C and G:

$$C_{tt'} = \langle\, \sigma[q(t), z(t)]\; \sigma[q(t'), z(t')]\, \rangle_\star, \tag{4.48}$$

$$G_{tt'} = -i \langle\, \sigma[q(t), z(t)]\; \hat{q}(t')\, \rangle_\star, \tag{4.49}$$

with

$$\langle g[\{q,\hat{q},z\}] \rangle_\star = \frac{\int \mathcal{D}q \mathcal{D}\hat{q}\, \langle M[\{q,\hat{q},z\}] g[\{q,\hat{q},z\}] \rangle_{\mathbf{z}}}{\int \mathcal{D}q \mathcal{D}\hat{q}\, \langle M[\{q,\hat{q},z\}] \rangle_{\mathbf{z}}}, \tag{4.50}$$

$$
\begin{aligned}
M[\{q,\hat{q},z\}] = p_0(q(0))\, &\mathrm{e}^{\mathrm{i} \sum_t \hat{q}(t)[q(t+1) - q(t) - \theta(t) + \alpha \sum_{t'} (\mathbf{1}+G)^{-1}_{tt'} \sigma[q(t'), z(t')]]} \\
&\times \mathrm{e}^{-\frac{1}{2}\alpha \sum_{tt'} \hat{q}(t) [(\mathbf{1}+G)^{-1} D (\mathbf{1}+G^\dagger)^{-1}]_{tt'} \hat{q}(t')}.
\end{aligned} \tag{4.51}
$$

4.3.3 The effective single-agent equation

Our final simplification consists of the elimination of the integration variables $\{\hat{q}(t)\}$, by exploiting the implications of causality. We note that $G_{tt'} = 0$ for $t \leq t'$ (since perturbations can only affect the future), so that the term $\sum_{t'} (\mathbf{1}+G)^{-1}_{tt'} \sigma[q(t'), z(t')] = \sum_{n \geq 0} (-1)^n (G^n)_{tt'} \sigma[q(t'), z(t')]$ in equation (4.51) involves only values of $q(t')$ with $t' \leq t$. This property will allow us to calculate the denominator of the fraction (4.50) by integrating out the variables $\{q(t)\}$ iteratively, first over $q(t_{\max})$ (which gives a term of the form $\delta[\hat{q}(t_{\max}-1)]$), followed by integration over $q(t_{\max}-1)$ (which gives a term of the form $\delta[\hat{q}(t_{\max}-2)]$), etc. All the powers of 2π hidden inside the shorthands $\mathcal{D}q\mathcal{D}\hat{q}$ are seen to be absorbed precisely by the emerging δ-functions, and the result is simply

$$\int \mathcal{D}q\mathcal{D}\hat{q}\; M[\{q,\hat{q},z\}] = \int dq(0)\; p_0(q(0)) = 1. \tag{4.52}$$

This, in turn, implies that

$$\langle \sigma[q(t),z(t)]\hat{q}(t')\rangle_\star = \mathrm{i}\frac{\partial}{\partial\theta(t')}\langle\sigma[q(t),z(t)]\rangle_\star.$$

We may now carry out the remaining integrations over $\{\hat{q}\}$, and write our saddle-point equations in the yet simpler form

$$C_{tt'} = \langle \sigma[q(t),z(t)]\sigma[q(t'),z(t')]\rangle_\star, \quad G_{tt'} = \frac{\partial\langle\sigma[q(t),z(t)]\rangle_\star}{\partial\theta(t')} \tag{4.53}$$

with $\langle g[\{q,z\}]\rangle_\star = \int[\prod_t \mathrm{d}q(t)]\;\langle M[\{q,z\}]g[\{q,z\}]\rangle_{\mathbf{z}}$, i.e. the effective measure $\langle\cdots\rangle_\star$ no longer involves a fraction, and

$$\begin{aligned}
M[\{q,z\}] &= p_0(q(0))\int\prod_t\left[\frac{\mathrm{d}\hat{q}(t)}{2\pi}\right]\mathrm{e}^{-\frac{1}{2}\alpha\sum_{tt'}\hat{q}(t)[(\mathbf{1}+G)^{-1}D(\mathbf{1}+G^\dagger)^{-1}]_{tt'}\hat{q}(t')} \\
&\quad\times\mathrm{e}^{\mathrm{i}\sum_t\hat{q}(t)[q(t+1)-q(t)-\theta(t)+\alpha\sum_{t'}(\mathbf{1}+G)^{-1}_{tt'}\sigma[q(t'),z(t')]]} \\
&= p_0(q(0))\int\prod_t\left[\frac{\mathrm{d}\eta(t)\mathrm{d}\hat{q}(t)}{2\pi}\right] \\
&\quad\times\mathrm{e}^{\mathrm{i}\sqrt{\alpha}\sum_t\hat{q}(t)\eta(t)-\frac{1}{2}\alpha\sum_{tt'}\hat{q}(t)[(\mathbf{1}+G)^{-1}D(\mathbf{1}+G^\dagger)^{-1}]_{tt'}\hat{q}(t')} \\
&\quad\times\prod_t\delta\left[\eta(t)-\frac{q(t+1)-q(t)-\theta(t)+\alpha\sum_{t'}(\mathbf{1}+G)^{-1}_{tt'}\sigma[q(t'),z(t')]}{\sqrt{\alpha}}\right] \\
&= p_0(q(0))\int\prod_t\left[\frac{\mathrm{d}\eta(t)}{\sqrt{2\pi}}\right]\frac{\mathrm{e}^{-\frac{1}{2}\sum_{tt'}\eta(t)[(\mathbf{1}+G)^{-1}D(\mathbf{1}+G^\dagger)^{-1}]_{tt'}\eta(t')}}{\sqrt{\det[(\mathbf{1}+G)^{-1}D(\mathbf{1}+G^\dagger)^{-1}]}} \\
&\quad\times\prod_{t\geq 0}\delta\left[q(t+1)-q(t)-\theta(t)+\alpha\sum_{t'}(\mathbf{1}+G)^{-1}_{tt'}\sigma[q(t'),z(t')]-\sqrt{\alpha}\eta(t)\right].
\end{aligned} \tag{4.54}$$

We recognize that the representation (4.54) is mathematically fully equivalent to the measure corresponding to a single-agent process of the form

$$q(t+1) = q(t) + \theta(t) - \alpha \sum_{t' \leq t} (\mathbf{1} + G)^{-1}_{tt'} \sigma[q(t'), z(t')] + \sqrt{\alpha}\, \eta(t) \quad (4.55)$$

in which $\eta(t)$ is a Gaussian noise, with zero mean and with temporal correlations given by $\langle \eta(t)\eta(t') \rangle = \Sigma_{tt'}$:

$$\Sigma = (\mathbf{1} + G)^{-1} D (\mathbf{1} + G^{\dagger})^{-1} \quad (4.56)$$

Causality ensures that $G_{tt'} = 0$ for all $t' \geq t$, so $(\mathbf{1}+G)^{-1}_{tt'} = \sum_{n \geq 0} (-1)^n [G^n]_{tt'} = 0$ for $t' > t$. Hence the third term in the right-hand side of equation (4.55) represents a retarded self-interaction, and therefore also the effective single-agent process (4.55) obeys causality. The correlation and response functions (4.14) and (4.15) are the dynamic order parameters of our problem, to be solved self-consistently from the closed equations (4.53), in which $\langle \cdots \rangle_\star$ now denotes averaging over the effective single-agent process (4.55) and over the zero-average Gaussian noise variables $\{z(t)\}$, with $\langle z(t) z(t') \rangle = \delta_{tt'}$. These results represent a fully exact and closed theory, describing both the statics and the dynamics of the MG for $N \to \infty$.

4.3.4 Overall bid statistics and volatility

Finally we would like to be able to express statistical properties of the overall bid such as the volatility, or more generally the volatility matrix (4.5), in terms of the solution $\{C, G\}$ of our effective single-trader problem (4.53) (4.55) and (4.56). It turns out that this requires only minor adaptations of our calculations so far. We define again a generating functional, in the spirit of equation (4.9):

$$Z[\boldsymbol{\phi}] = \langle e^{i\sqrt{2}\sum_{t \geq 0} \sum_\mu \phi_\mu(t) A^\mu[\mathbf{q}(t), \mathbf{z}(t)]} \rangle. \quad (4.57)$$

Here $\boldsymbol{\phi}$ denotes a new set of generating fields $\{\phi_i(t)\}$, and the definition of the average is identical to that in definition (4.9) (the extra factor $\sqrt{2}$ will give us somewhat cleaner equations later). From equation (4.57) one obtains

$$\langle A^\mu[\mathbf{q}(t), \mathbf{z}(t)] \rangle = -\frac{\mathrm{i}}{\sqrt{2}} \lim_{\boldsymbol{\phi} \to \mathbf{0}} \frac{\partial Z[\boldsymbol{\phi}]}{\partial \phi_\mu(t)}, \quad (4.58)$$

$$\langle A^\mu[\mathbf{q}(t), \mathbf{z}(t)] A^\nu[\mathbf{q}(t'), \mathbf{z}(t')] \rangle = -\frac{1}{2} \lim_{\boldsymbol{\phi} \to \mathbf{0}} \frac{\partial^2 Z[\boldsymbol{\phi}]}{\partial \phi_\mu(t) \partial \phi_\nu(t')}. \quad (4.59)$$

Close inspection of our previous calculation of $Z[\boldsymbol{\psi}]$ reveals that, provided we follow exactly the previous route of manipulations, we may find the desired expressions for

$Z[\boldsymbol{\phi}]$ and $\overline{Z[\boldsymbol{\phi}]}$ by making suitable replacements in the corresponding expressions for $Z[\boldsymbol{\psi}]$ and $\overline{Z[\boldsymbol{\psi}]}$. In particular, we find that in equation (4.17) we need just make the simple substitution

$$\mathrm{e}^{\mathrm{i}\sum_{it}\psi_i(t)s_i(t)} \;\longrightarrow\; \mathrm{e}^{\mathrm{i}\sum_{t\mu}\phi_\mu(t)[\sqrt{2}\Omega_\mu+x_t^\mu]}.$$

After a simple shift of the integration variables w_t^μ this then gives us

$$\begin{aligned} Z[\boldsymbol{\phi}] = &\int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\ \mathrm{e}^{\mathrm{i}\sum_{t\mu}[\hat{w}_t^\mu[w_t^\mu-\phi_\mu(t)]+\hat{x}_t^\mu x_t^\mu+w_t^\mu(\sqrt{2}\Omega_\mu+x_t^\mu)]} \\ &\times\int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}}\ p_0(\mathbf{q}(0))\ \langle \mathrm{e}^{\frac{-\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{\mu i}\xi_i^\mu\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]}\rangle_{\mathbf{z}} \\ &\times \mathrm{e}^{\mathrm{i}\sum_{ti}\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)]}. \end{aligned} \tag{4.60}$$

The disorder average (4.18) is seen not to be affected at all by our modification, so for initial conditions of the factorized form $p_0(\mathbf{q}) = \prod_i p_0(q_i)$, instead of equations (4.22) and (4.23) we now have

$$\begin{aligned} \overline{Z[\boldsymbol{\phi}]} = &\int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\ \mathrm{e}^{\mathrm{i}N\sum_{tt'}[\hat{C}_{tt'}C_{tt'}+\hat{K}_{tt'}K_{tt'}+\hat{L}_{tt'}L_{tt'}]+\mathcal{O}(N^0)} \\ &\times\int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\ \mathrm{e}^{\mathrm{i}\sum_{t\mu}[\hat{w}_t^\mu[w_t^\mu-\phi_\mu(t)]+\hat{x}_t^\mu x_t^\mu+w_t^\mu x_t^\mu]} \\ &\times \mathrm{e}^{-\frac{1}{2}\sum_{\mu tt'}\left[w_t^\mu w_{t'}^\mu+\hat{w}_t^\mu L_{tt'}\hat{w}_{t'}^\mu+2\hat{x}_t^\mu K_{tt'}\hat{w}_{t'}^\mu+\hat{x}_t^\mu C_{tt'}\hat{x}_{t'}^\mu\right]} \\ &\times\int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}}\ p_0(\mathbf{q}(0))\ \langle \mathrm{e}^{\mathrm{i}\sum_{ti}\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)]} \\ &\times \mathrm{e}^{-\mathrm{i}\sum_i\sum_{tt'}[\hat{C}_{tt'}s_i(t)s_i(t')+\hat{K}_{tt'}s_i(t)\hat{q}_i(t')+\hat{L}_{tt'}\hat{q}_i(t)\hat{q}_i(t')]}\rangle_{\mathbf{z}} \\ = &\int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\ \mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)} \end{aligned} \tag{4.61}$$

with

$$\Psi = \mathrm{i}\sum_{tt'}[\hat{C}_{tt'}C_{tt'} + \hat{K}_{tt'}K_{tt'} + \hat{L}_{tt'}L_{tt'}], \tag{4.62}$$

$$\begin{aligned} \Phi = &\frac{1}{N}\sum_\mu \log\left[\int \mathcal{D}w\mathcal{D}\hat{w}\mathcal{D}x\mathcal{D}\hat{x}\ \mathrm{e}^{\mathrm{i}\sum_t[\hat{w}_t[w_t-\phi_\mu(t)]+\hat{x}_t x_t+w_t x_t]}\right. \\ &\left.\times\ e^{-\frac{1}{2}\sum_{tt'}[w_t w_{t'}+\hat{w}_t L_{tt'}\hat{w}_{t'}+2\hat{x}_t K_{tt'}\hat{w}_{t'}+\hat{x}_t C_{tt'}\hat{x}_{t'}]}\right] \end{aligned} \tag{4.63}$$

$$\Omega = \frac{1}{N}\sum_i \log\Big\langle \int \mathcal{D}q\mathcal{D}\hat{q}\ p_0(q(0))$$

$$\times\ \mathrm{e}^{\mathrm{i}\sum_t \hat{q}(t)[q(t+1)-q(t)-\theta_i(t)]-i\sum_{tt'}\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')}$$
$$\times\ \left\langle \mathrm{e}^{-\mathrm{i}\sum_{tt'}[\hat{C}_{tt'}\sigma[q(t),z(t)]\sigma[q(t'),z(t')])+\hat{K}_{tt'}\sigma[q(t),z(t)]\hat{q}(t')]}\right\rangle_{\mathbf{z}} \tag{4.64}$$

Apart from the obvious absence of the previous generating fields $\boldsymbol{\psi}$ in equation (4.64) (which were ultimately put to zero in the previous calculations as well), the present equations differ from those which gave us the effective single-agent process (4.53) (4.55) and (4.56) only in the modification of the function Φ (4.63), which is seen to revert to its previous form (4.25) for $\boldsymbol{\phi} \to \mathbf{0}$. At the physical saddle-point we have $L = 0$ and $K = iG$, so with $D_{tt'} = 1 + C_{tt'}$ and upon integrating in (4.63) over $\{x_t\}$ we obtain

$$\begin{aligned}
\Phi &= \frac{1}{N}\sum_{\mu}\log\int \mathcal{D}w\mathcal{D}\hat{w}\ \mathrm{e}^{\mathrm{i}\sum_t \hat{w}_t[\sum_{t'}(\mathbf{1}+G^{\dagger})_{tt'}w_{t'}-\phi_\mu(t)]-\frac{1}{2}\sum_{tt'}w_t D_{tt'}w_{t'}} \\
&= -\alpha\log\det(\mathbf{1}+G^{\dagger}) + \frac{1}{N}\sum_{\mu}\log\int\prod_t\left[\frac{\mathrm{d}u_t\mathrm{d}\hat{w}(t)}{2\pi}\mathrm{e}^{\mathrm{i}\sum_t \hat{w}_t u_t}\right] \\
&\quad\times\ \mathrm{e}^{-\frac{1}{2}\sum_{tt'}[u_t+\phi_\mu(t)][(\mathbf{1}+G)^{-1}\boldsymbol{D}(\mathbf{1}+G^{\dagger})^{-1}]_{tt'}[u_{t'}+\phi_\mu(t')]} \\
&= -\alpha\log\det(\mathbf{1}+G^{\dagger}) \\
&\quad -\frac{1}{2N}\sum_{\mu}\sum_{tt'}\phi_\mu(t)[(\mathbf{1}+G)^{-1}D(\mathbf{1}+G^{\dagger})^{-1}]_{tt'}\phi_\mu(t').
\end{aligned} \tag{4.65}$$

We may now calculate the two derivatives required to work out the overall bid moments (4.58) and (4.59):

$$\frac{\partial N\Phi}{\partial\phi_\mu(t)} = -\sum_{t'}[(\mathbf{1}+G)^{-1}D(\mathbf{1}+G^{\dagger})^{-1}]_{tt'}\phi_\mu(t'), \tag{4.66}$$
$$\frac{\partial^2 N\Phi}{\partial\phi_\mu(t)\partial\phi_\nu(t')} = -\delta_{\mu\nu}[(\mathbf{1}+G)^{-1}\boldsymbol{D}(\mathbf{1}+G^{\dagger})^{-1}]_{tt'} \tag{4.67}$$

and find, with our previous shorthand $\{\mathcal{D}\} = \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}K\mathcal{D}\hat{K}\mathcal{D}L\mathcal{D}\hat{L}$ and using the normalizaton identity $Z[\mathbf{0}] = 1$,

$$\begin{aligned}
\lim_{N\to\infty}\overline{\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]\rangle} &= -\frac{\mathrm{i}}{\sqrt{2}}\lim_{N\to\infty}\lim_{\boldsymbol{\phi}\to\mathbf{0}}\frac{\int\{\mathcal{D}\}(\partial/\partial\phi_\mu(t))\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= -\frac{\mathrm{i}}{\sqrt{2}}\lim_{N\to\infty}\lim_{\boldsymbol{\phi}\to\mathbf{0}}\frac{\int\{\mathcal{D}\}\ \mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}(\partial N\Phi/\partial\phi_\mu(t))}{\int\{\mathcal{D}\}\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= 0,
\end{aligned} \tag{4.68}$$

$$
\begin{aligned}
&\lim_{N\to\infty} \overline{\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]A^\nu[\mathbf{q}(t'),\mathbf{z}(t')]\rangle} \\
&= -\frac{1}{2}\lim_{N\to\infty}\lim_{\boldsymbol{\phi}\to\mathbf{0}} \frac{\int\{\mathcal{D}\}\frac{\partial^2}{\partial\phi_\mu(t)\partial\phi_\nu(t')}\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\ e^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= -\frac{1}{2}\lim_{N\to\infty}\lim_{\boldsymbol{\phi}\to\mathbf{0}} \frac{\int\{\mathcal{D}\}\ \mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}\left[\frac{\partial^2 N\Phi}{\partial\phi_\mu(t)\partial\phi_\nu(t')}+\frac{\partial N\Phi}{\partial\phi_\mu(t)}\frac{\partial N\Phi}{\partial\phi_\nu(t')}\right]}{\int\{\mathcal{D}\}\ \mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\
&= \frac{1}{2}\delta_{\mu\nu}[(\mathbf{1}+G)^{-1}D(\mathbf{1}+G^\dagger)^{-1}]_{tt'}. \qquad (4.69)
\end{aligned}
$$

The disorder-averaged expectation value of the overall bid itself is thereby confirmed to be zero (as expected). Upon making the (safe) assumption that quantities like $p^{-1}\sum_{\mu=1}^{p}\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]\rangle$ must for $N\to\infty$ be self-averaging (i.e. lose their dependence on the microscopic realization of the disorder) we see furthermore that the disorder-averaged volatility matrix (4.5) is for $N\to\infty$ equal to

$$
\begin{aligned}
\overline{\Xi}_{tt'} &= \lim_{N\to\infty}\Bigg\{\frac{1}{p}\sum_{\mu=1}^{p}\overline{\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]A^\mu[\mathbf{q}(t'),\mathbf{z}(t')]\rangle} \\
&\qquad -\overline{\left[\frac{1}{p}\sum_{\mu=1}^{p}\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]\rangle\right]}\ \overline{\left[\frac{1}{p}\sum_{\nu=1}^{p}\langle A^\nu[\mathbf{q}(t'),\mathbf{z}(t')]\rangle\right]}\Bigg\} \\
&= \frac{1}{2}[(\mathbf{1}+G)^{-1}D(\mathbf{1}+G^\dagger)^{-1}]_{tt'} = \frac{1}{2}\Sigma_{tt'} \qquad (4.70)
\end{aligned}
$$

with the matrix Σ as defined in equation (4.56). Thus the noise term $\eta(t)$ in the effective single-agent process (4.55) represents the overall market fluctuations.

4.4 The first few time steps

It is quite remarkable that our problem of solving the batch MG dynamics has for $N\to\infty$ been reduced to the analysis of the effective single-trader equations (4.53)–(4.56), with the volatility and other statistical properties of the overall bid subsequently following from equation (4.70), without as yet having used *any* approximations. Stage two of the generating functional analysis approach is to try to solve the order parameters C and G from the trio of equations (4.53), (4.55), and (4.56). With the exception of the so-called spherical batch MG version (to be introduced in Chapter 7), this is generally non-trivial, and we will therefore have to restrict ourselves either to stationary state solutions, or to solving the first few time-steps of our process.

Especially in solving the first few time steps we will be able to rely heavily on the simplifications that follow from causality. For instance, we always have $(\mathbf{1}+G)^{-1}=\sum_{n\geq 0}(-1)^n G^n$, with causality enforcing

$$[G^n]_{tt'} = 0 \quad \text{for} \quad t' > t - n. \tag{4.71}$$

For simplicity we will regard the external fields (which indeed have so far never been included in our numerical simulations) only as infinitesimal perturbations, required for finding response functions but to be put to zero at the end of every calculation. We will also choose the simplest types of initial conditions, viz. $p_0(q(0)) = \delta[q(0) - q_0]$, and limit ourselves to the additive and multiplicative decision noise definitions (2.12) and (2.13), with $P(z) = (2\pi)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}z^2}$. In the extreme parameter limits $\alpha \to 0$ and $\alpha \to \infty$ we are able to go beyond just a few iteration steps, and solve (4.53), (4.55), and (4.56) for all times $t > 0$ that do not scale with α.

4.4.1 Finite time solutions for $\alpha \to \infty$

The easiest limit to handle is $\alpha \to \infty$, where we must expect to see a solution corresponding to effectively random decision making by our agents. Upon going back to (4.53), (4.55) and (4.56), one may confirm iteratively that for finite times $t > 0$ and in leading order in α our effective agent $q(t)$ will oscillate according to

$$\text{additive}: \quad q(t) = \alpha\, \mathrm{sgn}[q(0)](-1)^t + \mathcal{O}(\sqrt{\alpha}), \tag{4.72}$$

$$C_{tt'} = (-1)^{|t-t'|} + \mathcal{O}(\frac{1}{\sqrt{\alpha}}), \quad G_{tt'} = \mathcal{O}\left(\frac{1}{\sqrt{\alpha}}\right), \tag{4.73}$$

$$\overline{\Xi}_{tt'} = \frac{1}{2} + \frac{1}{2}(-1)^{|t-t'|} + \mathcal{O}\left(\frac{1}{\sqrt{\alpha}}\right), \tag{4.74}$$

$$\text{multiplicative}: \quad q(t) = \alpha\, \mathrm{sgn}[q(0)](-1)^{|t-t'|} \prod_{s=0}^{t-1} \mathrm{sgn}[1+Tz(s)] + \mathcal{O}(\sqrt{\alpha}), \tag{4.75}$$

$$C_{tt'} = (-\lambda(T))^{|t-t'|} + \mathcal{O}(\frac{1}{\sqrt{\alpha}}), \quad G_{tt'} = \mathcal{O}\left(\frac{1}{\sqrt{\alpha}}\right), \tag{4.76}$$

$$\overline{\Xi}_{tt'} = \frac{1}{2} + \frac{1}{2}(-\lambda(T))^{|t-t'|} + \mathcal{O}\left(\frac{1}{\sqrt{\alpha}}\right), \tag{4.77}$$

with $\lambda(T) = \mathrm{Erf}[1/T\sqrt{2}]$. Our system thus immediately enters a period-two oscillation, which is perturbed by noise in the case of equation (4.75), with the role of initial conditions reduced to determining the phase of this oscillation. In both cases we have $\lim_{\alpha\to\infty} \overline{\Xi}_{tt} = 1$, which confirms the $\sigma = 1$ bid fluctuation amplitude consistent with random trading, already from the start of the game. We also see that for additive noise the noise strength T has dropped out in leading order in α, whereas for multiplicative noise both the microscopic solution $q(t)$ and the resulting order parameters depend on T.

4.4.2 Finite time solutions for $\alpha \to 0$

As could have been expected, the limit $\alpha \to 0$ is more subtle. This is illustrated by the potential for the response function as well as the strength of the effective noise $\eta(t)$ to diverge when $q_0 \to 0$. We will return to the latter scenarios in more detail below. If for now we first restrict ourselves to initial conditions with $q_0 = \mathcal{O}(\alpha^0) \neq 0$, we find

$$q(t) = q_0 + \sum_{t'=0}^{t-1} \left[\theta(t') + \sqrt{\alpha}\,\eta(t')\right] + \mathcal{O}(\alpha) \tag{4.78}$$

with

$$\text{additive}: \quad C_{tt'} = \lambda^2(T/|q_0|) + \delta_{tt'}[1 - \lambda^2(T/|q_0|)] + \mathcal{O}(\sqrt{\alpha}), \tag{4.79}$$

$$G_{tt'} = \frac{\sqrt{2}}{T\sqrt{\pi}} \mathrm{e}^{-\frac{1}{2}q_0^2/T^2} + \mathcal{O}(\sqrt{\alpha}), \tag{4.80}$$

$$\text{multiplicative}: \quad C_{tt'} = \lambda^2(T) + \delta_{tt'}[1 - \lambda^2(T)] + \mathcal{O}(\sqrt{\alpha}), \tag{4.81}$$

$$G_{tt'} = \mathcal{O}(\sqrt{\alpha}), \tag{4.82}$$

One gains additional macroscopic information and insight into the microscopic state by working out the noise covariance matrix Σ (4.56). For both noise types we have $C_{tt'} = \delta_{tt'} + (1 - \delta_{tt'})c + \mathcal{O}(\sqrt{\alpha})$, with $c = \lambda^2(T/|q_0|)$ for additive noise and $c = \lambda^2(T)$ for multiplicative noise, so we may in both cases write

$$\eta(t) = \sum_{t' \leq t} (\mathbf{1} + G)^{-1}_{tt'} \left[z\sqrt{c} + \tilde{z}(t')\sqrt{1-c}\right] + \mathcal{O}(\sqrt{\alpha}) \tag{4.83}$$

with transformed zero-average Gaussian variables z and $\{\tilde{z}(t)\}$, which obey $\langle z^2 \rangle = 1$, $\langle z\tilde{z}(t)\rangle = 0$ and $\langle \tilde{z}(t)\tilde{z}(t')\rangle = \delta_{tt'}$.

In the case of additive decision noise we thus find, more explicitly

$$\eta(t) = z\lambda(T/|q_0|) \sum_{t' \leq t} (\mathbf{1} + G)^{-1}_{tt'} + \sqrt{1 - \lambda^2(T/|q_0|)} \sum_{t' \leq t} (\mathbf{1} + G)^{-1}_{tt'} \tilde{z}(t') + \mathcal{O}(\sqrt{\alpha}).$$

Upon introducing the shorthand $g = (\sqrt{2}/T\sqrt{\pi})\mathrm{e}^{-\frac{1}{2}q_0^2/T^2}$ we may write $G_{tt'} = g + \mathcal{O}(\sqrt{\alpha})$ for all $t > t'$ (with $G_{tt'} = 0$ for $t \leq t'$, due to causality). For such response functions one finds[17] that $\sum_{t' \leq t} (\mathbf{1} + G)^{-1}_{tt'} = (1-g)^t$, so

[17] To see this, one simply writes the sum to be calculated as $S_t = \sum_{t' \leq t} (\mathbf{1}+G)^{-1}_{tt'}$, one multiplies both sides of this identity by $(\mathbf{1}+G)_{st}$, and one performs a summation over t. This gives the equivalent problem $S_t = 1 - g\sum_{t'=0}^{t-1} S_{t'}$. The solution $S_t = (1-g)^t$ is trivially correct for $t = 1$, and easily verified to be correct for arbitrary integer $t > 1$ by induction.

$$\eta(t) = z(1-g)^t\lambda(T/|q_0|) + \sqrt{1-\lambda^2(T/|q_0|)}\sum_{t'\le t}(\mathbf{1}+G)^{-1}_{tt'}\tilde{z}(t') + \mathcal{O}(\sqrt{\alpha}). \quad (4.84)$$

For $\{\theta(t)\} \to 0$ we may now apparently sharpen and expand our previous statements (4.78)–(4.80) on the short-time solution for additive decision noise to

$$q(t) = q_0 + \sqrt{\alpha}\left\{\frac{z}{g}\left[1-(1-g)^t\right]\lambda(T/|q_0|) + \sqrt{1-\lambda^2(T/|q_0|)}\sum_{t'=0}^{t-1}\sum_{s=0}^{t'}(\mathbf{1}+G)^{-1}_{t's}\tilde{z}(s)\right\} + \mathcal{O}(\alpha), \quad (4.85)$$

$$C_{tt'} = \lambda^2(T/|q_0|) + \delta_{tt'}[1-\lambda^2(T/|q_0|)] + \mathcal{O}(\sqrt{\alpha}), \quad (4.86)$$

$$G_{tt'} = g + \mathcal{O}(\sqrt{\alpha}), \quad g = \frac{\sqrt{2}}{T\sqrt{\pi}}\mathrm{e}^{-\frac{1}{2}q_0^2/T^2}, \quad (4.87)$$

$$\Xi_{tt'} = (1-g)^{t+t'} \times \begin{cases} 1+\mathcal{O}(\sqrt{\alpha}) & \text{for } t=t' \\ \frac{1}{2}+\frac{1}{2}\lambda^2(T/|q_0|)+\mathcal{O}(\sqrt{\alpha}) & \text{for } t\neq t'. \end{cases} \quad (4.88)$$

The crucial impact of the initial conditions q_0 in the small α regime is immediately clear in equation (4.85). The key quantity is the short-time value $g \geq 0$ taken by the response function, which decreases monotonically with $|q_0|$. For sufficiently large values of $|q_0|$ we have $0 \leq g < 2$, and the oscillations are damped. For small values of $|q_0|$, on the other hand, we may have $g > 2$, leading to oscillations that grow exponentially in amplitude.

For multiplicative noise we no longer have a finite value for the response function, in leading order in α, so

$$\eta(t) = \lambda(T)z + \tilde{z}(t)\sqrt{1-\lambda^2(T)} + \mathcal{O}(\sqrt{\alpha}). \quad (4.89)$$

Thus for $\{\theta(t)\} \to 0$ our solution on finite times takes the form

$$q(t) = q_0 + \sqrt{\alpha}\left\{\lambda(T)zt + \sqrt{1-\lambda^2(T)}\sum_{t'=0}^{t-1}\tilde{z}(t')\right\} + \mathcal{O}(\alpha), \quad (4.90)$$

$$C_{tt'} = \lambda^2(T) + \delta_{tt'}[1-\lambda^2(T)] + \mathcal{O}(\sqrt{\alpha}), \quad (4.91)$$

$$G_{tt'} = \mathcal{O}(\sqrt{\alpha}) \quad (4.92)$$

$$\Xi_{tt'} = \begin{cases} 1+\mathcal{O}(\sqrt{\alpha}) & \text{for } t=t' \\ \frac{1}{2}[1+\lambda^2(T)]+\mathcal{O}(\sqrt{\alpha}) & \text{for } t\neq t'. \end{cases} \quad (4.93)$$

In contrast to the limit $\alpha \to \infty$, for small α and short times the differences between additive and multiplicative decision noise are profound (unless, of course, we put

$T \to 0$, where the two solutions above are indeed seen to become identical). For multiplicative noise there are no short-time oscillations whose amplitude and/or sign depend on q_0. In particular we find for the global bid fluctuations (of which the volatility is the time average)

$$\text{additive}: \quad \overline{\Xi}_{tt} = (1-g)^{2t} + \mathcal{O}(\sqrt{\alpha}) \tag{4.94}$$

$$\text{multiplicative}: \quad \overline{\Xi}_{tt} = 1 + \mathcal{O}(\sqrt{\alpha}). \tag{4.95}$$

4.4.3 Short times and intermediate α

For the first few time steps one can still calculate the order parameters for intermediate values of α explicitly, including the volatility. We recall that $D_{tt'} = 1 + C_{tt'}$ and that $C_{tt} = 1$ for any t. Furthermore, we note that expression (4.56) for the matrix Σ can be written as

$$\begin{aligned} \Sigma_{tt'} &= \sum_{n,n' \geq 0} (-1)^{n+n'} \sum_{s,s' \geq 0} [G^n]_{ts} D_{ss'} [G^{n'}]_{t's'} \\ &= \sum_{n=0}^{t} \sum_{n'=0}^{t'} (-1)^{n+n'} \sum_{s=0}^{t-n} \sum_{s'=0}^{t'-n'} [G^n]_{ts} D_{ss'} [G^{n'}]_{t's'}. \end{aligned} \tag{4.96}$$

This gives, for instance, for the first few iteration steps

$$\Sigma_{00} = D_{00} = 2, \tag{4.97}$$

$$\Sigma_{10} = D_{10} + G_{10} D_{00} = 1 + C_{10} - 2G_{10}, \tag{4.98}$$

$$\begin{aligned} \Sigma_{11} &= D_{11} - 2G_{10} D_{01} + (G_{10})^2 D_{00} \\ &= 2 - 2G_{10}(1 + C_{10}) + 2(G_{10})^2. \end{aligned} \tag{4.99}$$

We are now in a position to calculate the entries of all order parameter kernels that involve times $t, t' \leq 2$. Equation (4.55) tells us that

$$q(1) = q(0) + \theta(0) + \sqrt{\alpha}\, \eta(0) - \alpha\sigma[q(0), z(0)], \tag{4.100}$$

$$\begin{aligned} q(2) = {} & q(0) + \theta(0) + \theta(1) + \sqrt{\alpha}[\eta(0) + \eta(1)] \\ & -\alpha\sigma[q(0), z(0)] - \alpha\sigma[q(1), z(1)] + \alpha G_{10} \sigma[q(0), z(0)]. \end{aligned} \tag{4.101}$$

From these expressions we extract, in turn, using the previously introduced functions $\sigma[q] = \int \mathrm{d}z\, P(z) \sigma[q, z]$ and $\sigma'[q] = \mathrm{d}\sigma[q]/\mathrm{d}q$ (where $\sigma[-q] = -\sigma[q]$):

$$C_{10} = \left\langle \int \frac{\mathrm{d}\eta}{\sqrt{2\pi\Sigma_{00}}} \mathrm{e}^{-\frac{\eta^2}{2\Sigma_{00}}} \sigma[q_0 + \sqrt{\alpha}\, \eta - \alpha\sigma[q_0, z_0], z_1]\, \sigma[q_0, z_0] \right\rangle_{z_0, z_1}$$

$$
\begin{aligned}
&= \left\langle \int Dx\, \sigma\left[q_0 + x\sqrt{2\alpha} - \alpha\sigma[q_0, z_0]\right]\; \sigma[q_0, z_0] \right\rangle_{z_0} \\
&= \int DzDx\, \sigma\Big[(q_0 + x\sqrt{2\alpha})\sigma[q_0, z] - \alpha\Big], \qquad (4.102)
\end{aligned}
$$

$$
\begin{aligned}
G_{10} &= \lim_{\theta_0 \to 0} \frac{\partial}{\partial \theta_0} \left\langle \int \frac{\mathrm{d}\eta}{\sqrt{2\pi\Sigma_{00}}} e^{-\frac{\eta^2}{2\Sigma_{00}}} \sigma[q_0 + \theta_0 + \sqrt{\alpha}\,\eta - \alpha\sigma[q_0, z_0], z_1] \right\rangle_{z_0, z_1} \\
&= \int DzDx\, \sigma'\Big[(q_0 + x\sqrt{2\alpha})\sigma[q_0, z] - \alpha\Big]. \qquad (4.103)
\end{aligned}
$$

We first work out the above formulae for additive decision noise (2.12), i.e. for $\sigma[q, z] = \text{sgn}[q + Tz]$ and $\sigma[q] = \text{Erf}[q/T\sqrt{2}]$. Here we find, using integral (B.16) in Appendix B, that

$$
\begin{aligned}
C_{10} &= \int DzDx\, \text{Erf}\Big[\frac{(|q_0| + x\sqrt{2\alpha})\text{sgn}[z + |q_0|/T] - \alpha}{T\sqrt{2}}\Big] \\
&= -\frac{1}{2}\left[1 - \text{Erf}\Big[\frac{|q_0|}{T\sqrt{2}}\Big]\right] \int Dx\, \text{Erf}\Big[\frac{\alpha + |q_0| + x\sqrt{2\alpha}}{T\sqrt{2}}\Big] \\
&\quad -\frac{1}{2}\left[1 + \text{Erf}\Big[\frac{|q_0|}{T\sqrt{2}}\Big]\right] \int Dx\, \text{Erf}\Big[\frac{\alpha - |q_0| - x\sqrt{2\alpha}}{T\sqrt{2}}\Big] \\
&= -\frac{1}{2}\left[1 - \text{Erf}\Big[\frac{|q_0|}{T\sqrt{2}}\Big]\right] \text{Erf}\Big[\frac{\alpha + |q_0|}{\sqrt{2}\sqrt{T^2 + 2\alpha}}\Big] \\
&\quad -\frac{1}{2}\left[1 + \text{Erf}\Big[\frac{|q_0|}{T\sqrt{2}}\Big]\right] \text{Erf}\Big[\frac{\alpha - |q_0|}{\sqrt{2}\sqrt{T^2 + 2\alpha}}\Big], \qquad (4.104)
\end{aligned}
$$

$$
\begin{aligned}
G_{10} &= \frac{\sqrt{2}}{T\sqrt{\pi}} \int DzDx\, \mathrm{e}^{-\frac{1}{2}[(|q_0| + x\sqrt{2\alpha})\text{sgn}[z + |q_0|/T] - \alpha]^2/T^2} \\
&= \frac{1}{2}\left[1 - \text{Erf}\Big[\frac{|q_0|}{T\sqrt{2}}\Big]\right] \frac{\sqrt{2}\,\mathrm{e}^{-\frac{1}{2}(|q_0| + \alpha)^2/(T^2 + 2\alpha)}}{\sqrt{\pi(T^2 + 2\alpha)}} \\
&\quad + \frac{1}{2}\left[1 + \text{Erf}\Big[\frac{|q_0|}{T\sqrt{2}}\Big]\right] \frac{\sqrt{2}\,\mathrm{e}^{-\frac{1}{2}(|q_0| - \alpha)^2/(T^2 + 2\alpha)}}{\sqrt{\pi(T^2 + 2\alpha)}}. \qquad (4.105)
\end{aligned}
$$

In the limits $\alpha \to 0$ and $\alpha \to \infty$ we recover from these expressions our earlier results, in Sections 4.4.1 and 4.4.2. Let us now inspect these expressions (4.104) and (4.105) and the resulting values for the noise covariances (4.98) and (4.99) for the special case of 'tabula rasa' initial conditions (i.e. $q_0 \to 0$). Here one finds (4.104) and (4.105), and (4.98) and (4.99) reducing to

$$
C_{10} = -\text{Erf}\Big[\frac{\alpha}{\sqrt{2T^2 + 4\alpha}}\Big], \qquad (4.106)
$$

$$G_{10} = \frac{2\mathrm{e}^{-\alpha^2/(2T^2+4\alpha)}}{\sqrt{\pi(2T^2+4\alpha)}}, \tag{4.107}$$

$$\Sigma_{10} = 1 - \mathrm{Erf}\Big[\frac{\alpha}{\sqrt{2T^2+4\alpha}}\Big] - \frac{4\mathrm{e}^{-\alpha^2/(2T^2+4\alpha)}}{\sqrt{\pi(2T^2+4\alpha)}}, \tag{4.108}$$

$$\Sigma_{11} = 2 - \frac{4\mathrm{e}^{-\alpha^2/(2T^2+4\alpha)}}{\sqrt{\pi(2T^2+4\alpha)}}\Big\{1 - \mathrm{Erf}\Big[\frac{\alpha}{\sqrt{2T^2+4\alpha}}\Big]\Big\} + \frac{4\mathrm{e}^{-\alpha^2/(T^2+2\alpha)}}{\pi(T^2+2\alpha)}. \tag{4.109}$$

The negative value of C_{10} indicates that the effective agent initially starts alternating his strategies. In contrast to the situation where $q_0 \neq 0$ we also see that for 'tabula rasa' initialization and at $T = 0$ one will find a diverging value for the response G_{10} for small α, viz. $\lim_{T\to 0} G_{10} = (\pi\alpha)^{-\frac{1}{2}} + \mathcal{O}(\sqrt{\alpha})$.

Let us finally work out expressions (4.102) and (4.103) for multiplicative decision noise (2.13), i.e. for $\sigma[q,z] = \mathrm{sgn}[q]\mathrm{sgn}[1+Tz]$ and $\sigma[q] = \mathrm{sgn}[q]\lambda(T)$:

$$\begin{aligned} C_{10} &= \lambda(T)\int DzDx\ \mathrm{sgn}\Big[(|q_0| + x\sqrt{2\alpha})\mathrm{sgn}[1+Tz] - \alpha\Big] \\ &= -\lambda(T)\Big\{\frac{1}{2}[1-\lambda(T)]\mathrm{Erf}\Big[\frac{|q_0|+\alpha}{2\sqrt{\alpha}}\Big] + \frac{1}{2}[1+\lambda(T)]\mathrm{Erf}\Big[\frac{\alpha-|q_0|}{2\sqrt{\alpha}}\Big]\Big\}, \end{aligned} \tag{4.110}$$

$$\begin{aligned} G_{10} &= 2\lambda(T)\int DzDx\ \delta\Big[(|q_0| + x\sqrt{2\alpha})\mathrm{sgn}[1+Tz] - \alpha\Big] \\ &= \frac{\lambda(T)}{\sqrt{\alpha\pi}}\Big\{\frac{1}{2}[1-\lambda(T)]\mathrm{e}^{-(\alpha+|q_0|)^2/4\alpha} + \frac{1}{2}[1+\lambda(T)]\mathrm{e}^{-(\alpha-|q_0|)^2/4\alpha}\Big\}. \end{aligned} \tag{4.111}$$

For 'tabula rasa' initial conditions we find these formulae simplifying to

$$C_{10} = -\lambda(T)\mathrm{Erf}\left[\frac{1}{2}\sqrt{\alpha}\right], \tag{4.112}$$

$$G_{10} = \frac{\lambda(T)\mathrm{e}^{-\frac{1}{4}\alpha}}{\sqrt{\alpha\pi}}, \tag{4.113}$$

$$\Sigma_{10} = 1 - \lambda(T)\mathrm{Erf}\left[\frac{1}{2}\sqrt{\alpha}\right] - \frac{2\lambda(T)}{\sqrt{\alpha\pi}}\mathrm{e}^{-\frac{1}{4}\alpha}, \tag{4.114}$$

$$\Sigma_{11} = 2 - \frac{2\lambda(T)}{\sqrt{\alpha\pi}}\mathrm{e}^{-\frac{1}{4}\alpha}\Big\{1 - \lambda(T)\mathrm{Erf}[\frac{1}{2}\sqrt{\alpha}]\Big\} + \frac{2\lambda^2(T)}{\alpha\pi}\mathrm{e}^{-\frac{1}{2}\alpha}. \tag{4.115}$$

Comparison with the equivalent expressions (4.106)–(4.109) for additive decision noise leads to the conclusion that, in the small α regime, additive noise has a much stronger impact on the short-time dynamics of the batch MG, changing the leading order in α of order parameters such as C_{10} and G_{10} by a factor of $\sqrt{\alpha}$. In contrast, for multiplicative noise, raising the noise level from $T = 0$ to a non-zero value induces only a re-scaling of C_{10} and G_{10} by a factor $\lambda(T)$. We recall that in the pseudo-equilibrium replica calculation we could not make any statements on the behaviour of the system in the low α regime at all, except by ad hoc interpolation of the large α equations.

4.5 Stationary state in the ergodic regime

If the batch game has reached a time-translation invariant stationary state without long-term memory,[18] then $G_{tt'} = G(t-t')$ and $C_{tt'} = C(t-t')$. It now follows from equation (4.56) that also $\Sigma_{tt'} = \Sigma(t-t')$, and that therefore all three operators $\{C, G, \Sigma\}$ as well as their powers commute. In this section we assume that the stationary state is one without anomalous response, i.e. the integrated response $\chi = \lim_{\tau\to\infty} \sum_{t\leq\tau} G(t)$ is a finite non-negative number, and we will try to calculate persistent stationary state observables such as χ and $c = \lim_{t\to\infty} C(t)$ from the closed equations (4.53) (4.55) and (4.56). For such TTI states we may write

$$\begin{aligned}\sum_{t'} G^n(t-t') &= \sum_{t'} \Big[\sum_{t_1,\cdots,t_{n-1}} G(t-t_1)G(t_1-t_2)\cdots G(t_{n-2}-t_{n-1})G(t_{n-1}-t')\Big] \\ &= \chi \sum_{t_1,\cdots,t_{n-1}} G(t-t_1)G(t_1-t_2),\cdots,G(t_{n-2}-t_{n-1}) \\ &= \ \cdots \ = \chi^n. \end{aligned} \tag{4.116}$$

Hence also

$$\begin{aligned}\sum_{t'}(\mathbf{1}+G)^{-1}(t-t') &= \sum_{n\geq 0}(-1)^n \sum_{t'} G^n(t-t') \\ &= \sum_{n\geq 0}(-1)^n \chi^n = \frac{1}{1+\chi}. \end{aligned} \tag{4.117}$$

Given the desire not to introduce too many symbols, and given that the disorder averages are now well behind us, we will use the the following notation for time averages whenever there is no risk of ambiguity: $\overline{x} = \lim_{\tau\to\infty} \tau^{-1} \sum_{t=1}^{\tau} x(t)$.

4.5.1 Closed laws for persistent order parameters

In a stationary state of the original N-agent process (4.2) one generally finds agents who change strategy frequently (the 'fickle' agents), but also agents who do not ever change strategy (the 'frozen' agents). We expect the latter to correspond to values of

[18] Even if TTI solutions exist, it is not immediately obvious that they will necessarily be approached and reached (given that we have taken the limit $N \to \infty$ for finite t). In Chapter 7 we will deal with this issue in more detail.

$|q_i|$ that grow linearly in time. Our effective process (4.55) has replaced the original population statistics of equation (4.2) by the statistics of a single effective agent, so the original fraction ϕ of frozen agents has now become the fraction of 'effective agent' trajectories that end up growing linearly over time. It will turn out that in this section we no longer need the perturbation fields, so we put $\{\theta(t)\} \to 0$.

To separate effective single-agent trajectories $\{q(t)\}$ into fickle versus frozen ones, we introduce transformed variables $\tilde{q}(t) = q(t)/t$. The fraction of 'frozen' agents in the original N-agent system, for $N \to \infty$, will now be given by $\phi = \lim_{\epsilon\downarrow 0} \lim_{t\to\infty} \langle \theta[|\tilde{q}(t)| - \epsilon] \rangle_\star$. Carrying out this transformation in the process (4.55) gives the following equivalent equation in terms of $\{\tilde{q}(t)\}$:

$$\tilde{q}(t) = \frac{q(0)}{t} + \frac{\sqrt{\alpha}}{t} \sum_{t'=0}^{t-1} \eta(t') - \frac{\alpha}{t} \sum_{t'=0}^{t-1} \sum_{s\geq 0} (\mathbf{1}+G)^{-1}_{t's} \sigma[s\tilde{q}(s), z(s)]. \quad (4.118)$$

The tricky term in equation (4.118) is obviously the one that represents the retarded self-interaction. We may, however, use the property (4.117) to establish that, if our process indeed evolves towards a TTI state with $\chi < \infty$, and for all sequences $\{x(t)\}$ for which the limit $\lim_{\tau\to\infty} \tau^{-1} \sum_{t=0}^{\tau} x(t)$ exists

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \sum_{t'=0}^{t-1} \sum_{s\geq 0} (\mathbf{1}+G)^{-1}_{t's} x(s) &= \lim_{t\to\infty} \frac{1}{t} \sum_{t'=0}^{t-1} \sum_{s\geq 0} (\mathbf{1}+G)^{-1}(t'-s) x(s) \\
&= \sum_{s\geq 0} (\mathbf{1}+G)^{-1}(s) \lim_{t\to\infty} \frac{1}{t} \sum_{t'=s}^{t-1} x(t'-s) \\
&= \sum_{s\geq 0} (\mathbf{1}+G)^{-1}(s) \lim_{t\to\infty} (1-\frac{s}{t}) \frac{1}{t-s} \sum_{s'=0}^{t-1-s} x(s') \\
&= \overline{x} \sum_{s\geq 0} (\mathbf{1}+G)^{-1}(s) = \frac{\overline{x}}{1+\chi}. \qquad (4.119)
\end{aligned}$$

The crucial step was taken in the first line, where we used the fact that only those terms with $t' = \mathcal{O}(t)$, where time-translation invariance may be assumed to hold, can survive the limit $t \to \infty$). We now define $\tilde{q} = \lim_{t\to\infty} \tilde{q}(t)$, assuming that this limit exists, and send $t \to \infty$ in equation (4.118). The result is

$$\tilde{q} = \sqrt{\alpha}\, \overline{\eta} - \frac{\alpha \overline{\sigma}}{1+\chi}. \quad (4.120)$$

Here $\overline{\eta} = \lim_{\tau\to\infty} \tau^{-1} \sum_{t\leq\tau} \eta(t)$, and $\overline{\sigma} = \lim_{\tau\to\infty} \tau^{-1} \sum_{t\leq\tau} \sigma[\tilde{q}t]$. We note, given the general properties of the function $\sigma[q]$, that in those cases where $\tilde{q} \neq 0$ we must have

$\overline{\sigma} = \mathrm{sgn}[\tilde{q}].\sigma[\infty]$, whereas for $\tilde{q} = 0$ we know that $|\overline{\sigma}| \leq \sigma[\infty]$. We can now inspect the two possible solutions 'fickle' ($\tilde{q} = 0$) versus 'frozen' ($\tilde{q} \neq 0$), noting that in both cases we must have $\mathrm{sgn}[\overline{\sigma}] = \mathrm{sgn}[\overline{\eta}]$:

$$\text{'fickle'}: \quad \frac{\overline{\sigma}}{\sigma[\infty]} = \frac{(1+\chi)\overline{\eta}}{\sigma[\infty]\sqrt{\alpha}} \qquad \text{exists if } |\overline{\eta}| \leq \frac{\sigma[\infty]\sqrt{\alpha}}{1+\chi}, \tag{4.121}$$

$$\text{'frozen'}: \quad \mathrm{sgn}[\tilde{q}] = \frac{(1+\chi)[\overline{\eta} - \tilde{q}/\sqrt{\alpha}]}{\sigma[\infty]\sqrt{\alpha}} \quad \text{exists if } |\overline{\eta}| > \frac{\sigma[\infty]\sqrt{\alpha}}{1+\chi}. \tag{4.122}$$

We make the crucial observation that the two existence conditions in equations (4.121) and (4.122) are complementary.[19] Hence, which solution type of (4.120) one will find depends *solely* on the value of the zero-average Gaussian variable $\overline{\eta}$. This allows us to invert the above statements and write directly

$$|\overline{\eta}| \leq \frac{\sigma[\infty]\sqrt{\alpha}}{1+\chi}: \quad \text{'fickle' solution,} \quad \overline{\sigma} = \frac{(1+\chi)\overline{\eta}}{\sqrt{\alpha}}, \tag{4.123}$$

$$|\overline{\eta}| > \frac{\sigma[\infty]\sqrt{\alpha}}{1+\chi}: \quad \text{'frozen' solution,} \quad \overline{\sigma} = \sigma[\infty]\mathrm{sgn}[\overline{\eta}]. \tag{4.124}$$

The explicit and convenient classification provided by equations (4.123) and (4.124) will be our main tool in deriving closed stationary state equations. The remaining programme of work is: (i) calculating the variance of the zero-average Gaussian variable $\overline{\eta}$ from equation (4.56), (ii) calculating the persistent correlations $c = \lim_{t\to\infty} C(t)$ from equations (4.123) and (4.124), (iii) calculating the frozen fraction ϕ from (4.123) and (4.124), and (iv) finding an expression for the susceptibility χ.

- The variance of $\overline{\eta}$ follows directly from equation (4.56):

$$\begin{aligned}
\langle \overline{\eta}^2 \rangle &= \lim_{\tau\to\infty} \frac{1}{\tau^2} \sum_{tt'=0}^{\tau} [(\mathbf{1}+G)^{-1} D (\mathbf{1}+G^\dagger)^{-1}]_{tt'} \\
&= \sum_{ss'} (\mathbf{1}+G)^{-1}(s)(\mathbf{1}+G)^{-1}(s') \lim_{\tau\to\infty} \frac{1}{\tau^2} \sum_{t=s}^{\tau} \sum_{t'=s'}^{\tau} [1+C(t-s-t'+s')] \\
&= \frac{1+c}{(1+\chi)^2}.
\end{aligned} \tag{4.125}$$

[19] Here we have used the assumption that $1+\chi > 0$, which seems physically natural in view of the definition of the response function. However, in contrast to equilibrium calculations, in the present dynamical formalism one cannot rule out solutions with $1+\chi < 0$ purely on mathematical grounds. Such solutions would describe a situation where the retarded self-interaction term in the effective single-agent equation (4.55) has switched from acting as a negative feedback to a positive feedback, and would generate remanence effects in our process.

- The persistent correlations can be expressed in terms of the effective single-agent via $c = \lim_{\tau\to\infty} \tau^{-2}\sum_{tt'\leq\tau}\langle\sigma[q(t),z(t)]\sigma[q(t'),z(t')]\rangle_\star = \langle\overline{\sigma}^2\rangle_\star$. At this stage we must distinguish between the two cases in equations (4.123) and (4.124):

$$
\begin{aligned}
c &= \int \mathrm{d}\overline{\eta}\, P(\overline{\eta}) \left\{\theta\Big[\frac{\sigma[\infty]\sqrt{\alpha}}{1+\chi} - |\overline{\eta}|\Big]\frac{(1+\chi)^2\overline{\eta}^2}{\alpha} + \theta\Big[|\overline{\eta}| - \frac{\sigma[\infty]\sqrt{\alpha}}{1+\chi}\Big]\sigma^2[\infty]\right\} \\
&= \frac{2(1+c)}{\alpha}\int_0^\infty Dz\, \theta\Big[\frac{\sigma[\infty]\sqrt{\alpha}}{\sqrt{1+c}} - z\Big] z^2 + 2\sigma^2[\infty]\int_0^\infty Dz\, \theta\Big[z - \frac{\sigma[\infty]\sqrt{\alpha}}{\sqrt{1+c}}\Big] \\
&= \frac{1+c}{\alpha}\left\{\mathrm{Erf}\Big[\frac{\sigma[\infty]\sqrt{\alpha}}{\sqrt{2(1+c)}}\Big] - \sqrt{\frac{2}{\pi}}\frac{\sigma[\infty]\sqrt{\alpha}}{\sqrt{1+c}}\mathrm{e}^{-\frac{\sigma^2[\infty]\alpha}{2(1+c)}}\right\} \\
&\quad + \sigma^2[\infty]\left\{1 - \mathrm{Erf}\Big[\frac{\sigma[\infty]\sqrt{\alpha}}{\sqrt{2(1+c)}}\Big]\right\}. \qquad (4.126)
\end{aligned}
$$

- Given our conditions (4.123) and (4.124), the frozen fraction ϕ can be calculated similarly. Here we have

$$
\begin{aligned}
\phi &= \int \mathrm{d}\overline{\eta}\, P(\overline{\eta})\, \theta\Big[|\overline{\eta}| - \frac{\sigma[\infty]\sqrt{\alpha}}{1+\chi}\Big] = 2\int_0^\infty Dz\, \theta\Big[z - \frac{\sigma[\infty]\sqrt{\alpha}}{\sqrt{1+c}}\Big] \\
&= 1 - \mathrm{Erf}\Big[\frac{\sigma[\infty]\sqrt{\alpha}}{\sqrt{2(1+c)}}\Big]. \qquad (4.127)
\end{aligned}
$$

- To find the susceptibility $\chi = \lim_{\tau\to\infty}\sum_{t\leq\tau} G(t)$ we note that it is identical to the asymptotic response of the system to a *time-independent* external perturbation, viz. $\theta(t) = \theta$ for all t:

$$
\begin{aligned}
\chi &= \lim_{t\to\infty}\frac{\partial}{\partial\theta}\langle\sigma[q(t),z(t)]\rangle_\star = \frac{1}{\sqrt{\alpha}}\langle\frac{\partial}{\partial\overline{\eta}}\overline{\sigma}\rangle_\star \\
&= \frac{1}{\sqrt{\alpha}}\int \mathrm{d}\overline{\eta}\, P(\overline{\eta})\frac{\partial\overline{\sigma}}{\partial\overline{\eta}} = \frac{(1+\chi)^2}{(1+c)\sqrt{\alpha}}\int \mathrm{d}\overline{\eta}\, P(\overline{\eta})\,\overline{\eta}\,\overline{\sigma} \\
&= \frac{1+\chi}{\sqrt{\alpha(1+c)}}\int Dz\,\frac{\partial}{\partial z}\left\{\theta\Big[\frac{\sigma[\infty]\sqrt{\alpha}}{\sqrt{1+c}} - |z|\Big]\frac{z\sqrt{1+c}}{\sqrt{\alpha}}\right. \\
&\quad \left. + \theta\Big[|z| - \frac{\sigma[\infty]\sqrt{\alpha}}{\sqrt{1+c}}\Big]\sigma[\infty]\mathrm{sgn}[z]\right\} \\
&= \left(\frac{1+\chi}{\alpha}\right)\mathrm{Erf}\left[\frac{\sigma[\infty]\sqrt{\alpha}}{\sqrt{2(1+c)}}\right]. \qquad (4.128)
\end{aligned}
$$

This gives, together with equation (4.127):

$$
\chi = \frac{1-\phi}{\alpha - 1 + \phi}. \qquad (4.129)
$$

We have now arrived at an exact and fully closed set of equations involving only the persistent scalar order parameters $\{c, \phi, \chi\}$ (given the assumptions of time-translation invariance and finite integrated response $\chi > -1$):

$$c = \sigma^2[\infty]\left\{1 + \frac{1-2v^2}{2v^2}\mathrm{Erf}[v] - \frac{1}{v\sqrt{\pi}}\mathrm{e}^{-v^2}\right\}, \tag{4.130}$$

$$\phi = 1 - \mathrm{Erf}[v], \tag{4.131}$$

$$\chi = \frac{\mathrm{Erf}[v]}{\alpha - \mathrm{Erf}[v]}, \tag{4.132}$$

with the shorthand $v = \sigma[\infty]\sqrt{\alpha/2(1+c)}$.

We see already in equation (4.129), as was found earlier in the approximate replica calculation in Chapter 3, that the phase transition marked by a divergence of χ occurs when the relation $\phi = 1 - \alpha$ holds. We now know this intriguing statement to be exact for the batch MG, irrespective of either type or strength of the decision noise. Insertion of this condition into equations (4.130) and (4.131) shows that the transition line in the (α, T) plane is given by $\alpha_{\mathrm{c}}(T) = \mathrm{Erf}[v_{\mathrm{c}}(T)]$, where $v_{\mathrm{c}}(T)$ is the solution of the transcendental equation

$$\sigma^2[\infty]\left\{\mathrm{Erf}[v] - 1 + \frac{1}{v\sqrt{\pi}}\mathrm{e}^{-v^2}\right\} = 1. \tag{4.133}$$

4.5.2 Additive decision noise

In the case of (Gaussian) additive decision noise (2.12) we simply have $\sigma[\infty] = 1$, and we immediately recover from equations (4.130)–(4.132) within the present (exact) generating functional analysis formalism the stationary state solution that we had found earlier using the pseudo-equilibrium analysis based on the replica method

$$c = 1 + \frac{1-2v^2}{2v^2}\mathrm{Erf}[v] - \frac{1}{v\sqrt{\pi}}\mathrm{e}^{-v^2}, \tag{4.134}$$

$$\phi = 1 - \mathrm{Erf}[v], \quad \chi = \frac{\mathrm{Erf}[v]}{\alpha - \mathrm{Erf}[v]}, \quad v = \frac{\sqrt{\alpha}}{\sqrt{2(1+c)}} \tag{4.135}$$

(apart from a simple re-scaling by a factor α of the susceptibility χ, which is caused by the different definition of time units in Chapter 3). The role of the RS order parameter q is now seen to be played by the persistent correlation c. We note again that the noise parameter T has dropped out completely, i.e. additive decision noise is once more predicted to be of no consequence in the ergodic stationary state. The phase transition line is independent of T, and given by $\alpha_c = \text{Erf}[v_c]$, where v_c is the solution of

$$\text{Erf}[v] + \frac{1}{v\sqrt{\pi}}e^{-v^2} = 2. \tag{4.136}$$

Numerical solution gives, as noted earlier, $\alpha_c \approx 0.33740$.

The present solution formally applies only to the regime $\alpha > \alpha_c$, where $0 < 1+\chi < \infty$ as required. Its natural (ad hoc) extension into the regime $\alpha < \alpha_c$, following the logic applied earlier with success in the pseudo-equilibrium analysis chapter, would be to continue with $\chi = \infty$ and sacrifice the identity $v = \sqrt{\alpha}/\sqrt{2(1+c)}$. The result is

$$\alpha < \alpha_c : \quad c = 1 - \alpha + \frac{\alpha}{2v^2} - \frac{1}{v\sqrt{\pi}}e^{-v^2}, \tag{4.137}$$

$$\phi = 1 - \alpha, \qquad \text{Erf}[v] = \alpha \tag{4.138}$$

In Figs 4.1 and 4.2, referring to the noise levels $T = 0$ and $T = 1$, respectively, we show how the predictions for ϕ and c (based on numerical solution of the coupled equations (4.134) and (4.135) and their ad hoc extension (4.137) and (4.138) into the non-ergodic regime) compare with the results of carrying out numerical simulations of the batch MG with additive decision noise. One has to be careful in such simulations, since additive decision noise has a non-negligible effect on the relaxation times in the

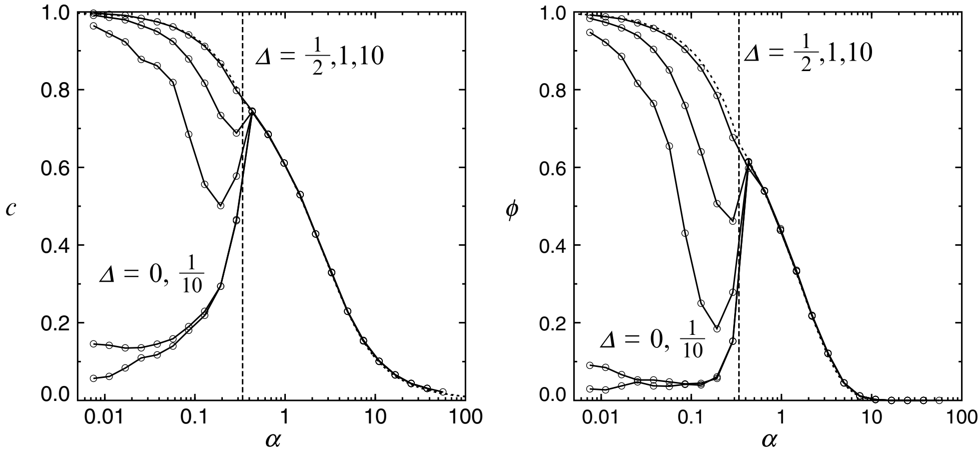

Fig. 4.1 Left: the predicted persistent correlations $c = \lim_{t\to\pm\infty} C(t)$ (dotted curve), together with the simulation data (connected markers), for the fake memory batch MG without decision noise. Right: the predicted fraction ϕ of frozen agents (dotted curve), together with the simulation data (connected markers). Initial conditions: $|q_i(0)| = \Delta$ for all i, with $\Delta \in \{0,\ 0.1,\ 0.5,\ 1,\ 10\}$ (from bottom to top in both pictures). The vertical dashed line marks the predicted critical value $\alpha_c \approx 0.33740$.

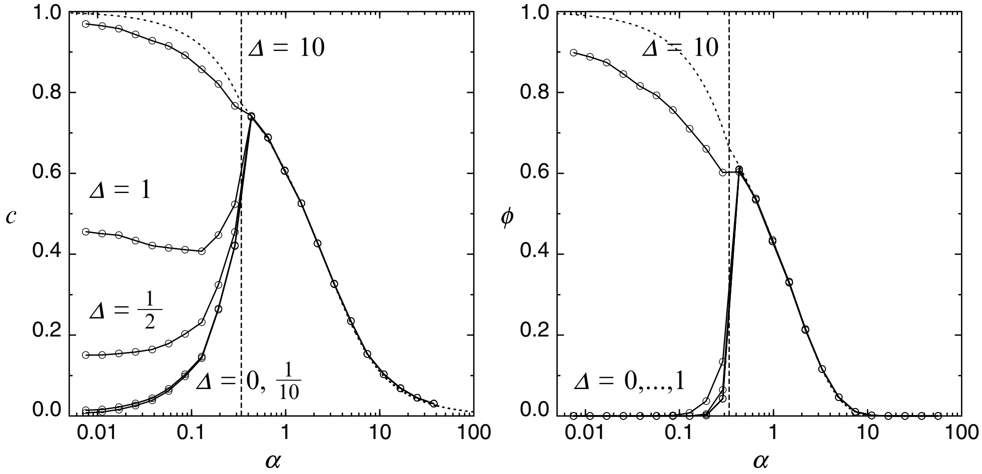

Fig. 4.2 Left: the predicted persistent correlations $c = \lim_{t\to\pm\infty} C(t)$ (dotted curve), together with the simulation data (connected markers), for the fake memory batch MG with additive decision noise of strength $T = 1$. Right: the predicted fraction ϕ of frozen agents (dotted curve), together with the simulation data (connected markers). Initial conditions: $|q_i(0)| = \Delta$ for all i, with $\Delta \in \{0,\ 0.1,\ 0.5,\ 1,\ 10\}$ (from bottom to top in both pictures). The vertical dashed line marks the predicted critical value $\alpha_c \approx 0.33740$.

MG; for $T > 0$ it is very easy to find spurious results caused simply by insufficient equilibration. It will be clear from our figures that the present equations are correct only in their proper regime of validity, i.e. in the ergodic phase $\alpha > \alpha_c$. Below the transition point, for small α, the macroscopic observables of the batch MG do depend strongly on the strength of the decision noise (which is consistent with our earlier observations in solving the first few time steps of the dynamics).

4.5.3 Multiplicative decision noise

In the case of (Gaussian) multiplicative decision noise (2.13), in contrast, we have $\sigma[\infty] = \lambda(T) = \text{Erf}[1/T\sqrt{2}]$, and the noise does play an important role at the macroscopic level, even in the ergodic regime. Again we recover also for multiplicative noise from (4.130)–(4.132) the stationary state solution which we had found earlier using the replica method (modulo the re-scaling of χ)

$$c = \lambda^2(T)\left\{1 + \frac{1-2v^2}{2v^2}\text{Erf}[v] - \frac{1}{v\sqrt{\pi}}\text{e}^{-v^2}\right\}, \tag{4.139}$$

$$\phi = 1 - \text{Erf}[v], \quad \chi = \frac{\text{Erf}[v]}{\alpha - \text{Erf}[v]}, \quad v = \frac{\sqrt{\alpha}\lambda(T)}{\sqrt{2(1+c)}}. \tag{4.140}$$

Now the phase transition line does depend on T, and is given by $\alpha_c(T) = \text{Erf}[v_c(T)]$, where $v_c(T)$ is the solution of

$$\lambda^2(T)\left\{\text{Erf}[v] - 1 + \frac{1}{v\sqrt{\pi}}\text{e}^{-v^2}\right\} = 1. \tag{4.141}$$

This result was found earlier with the pseudo-equilibrum analysis, and implies the phase diagram shown in Fig. 3.4. We can now be confident that for the batch MG these early results are exact.

Again the above solution applies only to the regime $\alpha > \alpha_c(T)$, due to the requirements $0 < 1 + \chi < \infty$. Its (ad hoc) extension into the regime $\alpha < \alpha_c(T)$, based on the assumption that $\chi = \infty$ throughout, now becomes

$$\alpha < \alpha_c: \quad c = \lambda^2(T)\left\{1 - \alpha + \frac{\alpha}{2v^2}\text{Erf}[v] - \frac{1}{v\sqrt{\pi}}\text{e}^{-v^2}\right\} \tag{4.142}$$

$$\phi = 1 - \alpha, \qquad \text{Erf}[v] = \alpha. \tag{4.143}$$

In Fig. 4.3 we show the resulting predictions for the persistent observables ϕ and c (based on numerical solution of the coupled equations (4.139) and (4.140), and their ad hoc extension (4.142) and (4.143) into the non-ergodic regime), compared with

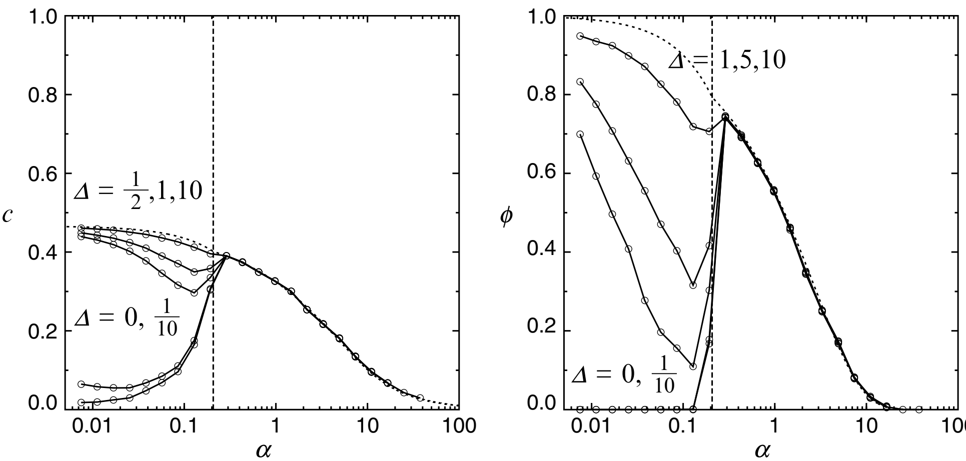

Fig. 4.3 Left: the predicted persistent correlations $c = \lim_{t\to\pm\infty} C(t)$ (dotted curve), together with the simulation data (connected markers), for the fake memory batch MG with multiplicative decision noise of strength $T = 1$. Right: the predicted fraction ϕ of frozen agents (dotted curve), together with the simulation data (connected markers). Initial conditions: $|q_i(0)| = \varDelta$ for all i, with $\varDelta \in \{0,\ 0.1,\ 0.5,\ 1,\ 10\}$ (from bottom to top in both pictures). The vertical dashed line marks the predicted critical value $\alpha_c(1) \approx 0.206930$.

the results of carrying out numerical simulations of the batch MG with multiplicative decision noise, of strength $T = 1$. We see again that the equations for our persistent order parameters work perfectly in the regime where they are supposed to, i.e. for $\alpha > \alpha_c(T)$.

4.6 Volatility in the ergodic regime

4.6.1 The exact expression

Although also in the generating functional approach the volatility is not itself an order parameter, here we have the considerable advantage that we can express it exactly in terms of our (dynamic) order parameters, via equation (4.70):

$$\sigma^2 = \lim_{\tau\to\infty} \frac{1}{2\tau} \sum_{t\leq\tau} [(\mathbf{1}+G)^{-1} D (\mathbf{1}+G^{\dagger})^{-1}]_{tt} \tag{4.144}$$

In TTI states, and using $C(-s) = C(s)$ as well as equation (4.117), this formula can be simplified. We separate the correlations into persistent and non-persistent contributions, viz. $C(t) = c + \tilde{C}(t)$, where $\lim_{t\to\infty} \tilde{C}(t) = 0$, and find

$$\begin{aligned}
\sigma^2 &= \lim_{\tau\to\infty} \frac{1}{2\tau} \sum_{t\leq\tau} \sum_{ss'} (\mathbf{1}+G)^{-1}(t-s)[1 + C(s-s')](\mathbf{1}+G)^{-1}(t-s') \\
&= \frac{1}{2} \sum_{ss'} (\mathbf{1}+G)^{-1}(s)[1 + C(s-s')](\mathbf{1}+G)^{-1}(s') \\
&= \frac{1+c}{2(1+\chi)^2} + \frac{1}{2} \sum_{ss'} (\mathbf{1}+G)^{-1}(s) \tilde{C}(s-s') (\mathbf{1}+G)^{-1}(s').
\end{aligned} \tag{4.145}$$

This expression is still fully exact, but it will be clear that we cannot hope to find a similarly exact statement involving only the persistent order parameters $\{c, \phi, \chi\}$. The volatility involves short-term fluctuations, via $\tilde{C}(t)$, which can only be neglected at the cost of approximation.

The only fluctuation measure that can be expressed in terms of persistent order parameters only, without approximations, appears to be the following quantity, which measures the global bid correlations at widely separated times

$$\begin{aligned}
\Xi(\infty) = \lim_{\tau\to\infty} \lim_{N\to\infty} \Bigg\{ & \frac{1}{p} \sum_{\mu=1}^{p} \overline{\langle A^\mu[\mathbf{q}(t+\tau), \mathbf{z}(t+\tau)] A^\mu[\mathbf{q}(t), \mathbf{z}(t)] \rangle} \\
& - \overline{\Big[\frac{1}{p} \sum_{\mu=1}^{p} \langle A^\mu[\mathbf{q}(t+\tau), \mathbf{z}(t+\tau)] \rangle\Big]}\ \overline{\Big[\frac{1}{p} \sum_{\nu=1}^{p} \langle A^\nu[\mathbf{q}(t), \mathbf{z}(t)] \rangle\Big]} \Bigg\}
\end{aligned}$$

$$= \frac{1+c}{2(1+\chi)^2}. \tag{4.146}$$

The fact that this latter quantity is nonzero for $\alpha > \alpha_c$ implies that in the ergodic regime of the batch MG there is potential for global bid prediction.

4.6.2 Approximation in terms of persistent objects

Although there is no objection in principle to working with the exact formula (4.145), in practice there would be obvious benefit in not having to solve our order parameters $\{C, G\}$ for all times. Thus we might wish to find a decent approximation of equation (4.145) in terms of the persistent quantities $\{c, \phi, \chi\}$ only, which will always be a somewhat messy exercise compared with the rigour that our generating functional formalism has allowed for so far.

The conventional route towards such approximations is to concentrate on the correlation function in equation (4.145). We obviously have $C(0) = 1$ and $\tilde{C}(0) = 1 - c$; the problem lies in the values of $\tilde{C}(t)$ for $t \neq 0$. Let us work out a number of possible ways of dealing with the latter:

- An approximation similar to that obtained earlier in the context of the replica calculation, viz. (3.68) (with the role of the replica order parameter q now being played by the persistent correlations c), can be obtained from equation (4.145) by just putting $\tilde{C}(t) \to 0$ for $t \neq 0$, giving

$$\sigma_A^2 = \frac{1+c}{2(1+\chi)^2} + \frac{1}{2}(1-c)V, \tag{4.147}$$

$$V = [(\mathbf{1}+G)^{-1}(\mathbf{1}+G^\dagger)^{-1}](0). \tag{4.148}$$

  This still leaves us with the task to evaluate or approximate the operator product (4.148). We will come back to this intermediate result below.

- Let us first turn to an alternative approximation. For $t > t'$ we may average in $\langle \sigma[q(t), z(t)]\sigma[q(t'), z(t')]\rangle_\star$ directly over the Gaussian variable $z(t)$, and subsequently separate these correlations into a 'frozen' and a 'fickle' contribution

$$C(t-t') = \phi\langle \sigma[q(t)]\sigma[q(t'), z(t')]\rangle_{\text{frozen}} + (1-\phi)\langle \sigma[q(t)]\sigma[q(t'), z(t')]\rangle_{\text{fickle}}.$$

  Here $\langle \cdots \rangle_{\text{fickle}}$ and $\langle \cdots \rangle_{\text{frozen}}$ are conditional averages over the effective agent process, with the defining conditions referring to the realization of the persistent effective noise $\overline{\eta}$, following equations (4.123) and (4.124). For frozen agents we obviously have

$$\langle\sigma[q(t)]\sigma[q(t'),z(t')]\rangle_{\text{frozen}}=\sigma^2[\infty]. \tag{4.149}$$

Thus we may write, without approximations as yet

$$\begin{aligned}\sigma^2&=\frac{1}{2(1+\chi)^2}+\frac{1}{2}V+\frac{1}{2}\phi\sigma^2[\infty]\sum_{s\neq s'}(\mathbf{1}+G)^{-1}(s)(\mathbf{1}+G)^{-1}(s')\\&\quad+\frac{1}{2}(1-\phi)\sum_{s\neq s'}(\mathbf{1}+G)^{-1}(s)(\mathbf{1}+G)^{-1}(s')\langle\sigma[q(s)]\sigma[q(s'),z(s')]\rangle_{\text{fickle}}\\&=\frac{1+\phi\,\sigma^2[\infty]}{2(1+\chi)^2}+\frac{1}{2}(1-\phi\,\sigma^2[\infty])V\\&\quad+\frac{1}{2}(1-\phi)\sum_{s\neq s'}(\mathbf{1}+G)^{-1}(s)(\mathbf{1}+G)^{-1}(s')\langle\sigma[q(s)]\sigma[q(s'),z(s')]\rangle_{\text{fickle}}.\end{aligned} \tag{4.150}$$

There are two obstacles left: the 'fickle' correlations in the last line, and (again) the operator product (4.148). We will simply assume the fickle co-variances at different times to vanish (in effect treating the fickle terms as pseudo-random ones), giving us an alternative approximation

$$\sigma_B^2=\frac{1+\phi\,\sigma^2[\infty]}{2(1+\chi)^2}+\frac{1}{2}(1-\phi\,\sigma^2[\infty])V. \tag{4.151}$$

- In our third approximation we return to the as yet exact stage (4.150), but now deal with the fickle correlations differently. Here we write

$$\begin{aligned}&\sum_{s\neq s'}(\mathbf{1}+G)^{-1}(s)(\mathbf{1}+G)^{-1}(s')\langle\sigma[q(s)]\sigma[q(s'),z(s')]\rangle_{\text{fickle}}\\&\quad=\left\langle\Big[\sum_s(\mathbf{1}+G)^{-1}(s)\sigma[q(s),z(s)]\Big]^2\right\rangle_{\text{fickle}}-\sum_s\Big[(\mathbf{1}+G)^{-1}(s)\Big]^2\\&\quad=\lim_{t\to\infty}\left\langle\Big[\sum_{t'\leq t}(\mathbf{1}+G)^{-1}_{tt'}\sigma[q(t'),z(t')]\Big]^2\right\rangle_{\text{fickle}}-V.\end{aligned} \tag{4.152}$$

We recognize the summation over t' inside the first term as the retarded self-interaction of our effective single-agent process (4.55). For 'fickle' solutions one should expect this term, which in disordered systems is usually responsible for memory effects and slow dynamics, to be absent (apart from the instantaneous contribution $t'=t$). We may therefore replace $\sum_{t'\leq t}\ldots$ simply by $\sigma[q(t),z(t)]$, and find equation (4.150) reducing to

$$\sigma_C^2=\frac{1+\phi\,\sigma^2[\infty]}{2(1+\chi)^2}+\frac{1}{2}(1-\phi\,\sigma^2[\infty])V+\frac{1}{2}(1-\phi)(1-V)$$

$$= \frac{1+\phi\,\sigma^2[\infty]}{2(1+\chi)^2} + \frac{1}{2}(1-\phi) + \frac{1}{2}\phi(1-\sigma^2[\infty])V. \tag{4.153}$$

In all three approximations (4.147) (4.151) and (4.153) we are still left with the task to evaluate or approximate equation (4.148), except for the case of additive (or absent) decision noise, where $\sigma^2[\infty] = 1$ and V is seen to drop out of equation (4.153).

A rigorous calculation of (4.148) would still involve knowing our time dependent order parameters for intermediate times, whereas the whole point of our present efforts is to find ways of avoiding this. We are therefore forced to make a final step: given the intuitively very similar assumptions that went into approximations (4.151) and (4.153) we demand that the results (4.151) and (4.153) must be identical.[20] This immediately leads us to the conclusion that $V = 1$, and we end up with just two distinct approximations for the volatility in terms of persistent quantities, namely

$$\sigma_A^2 = \frac{1+c}{2(1+\chi)^2} + \frac{1}{2}(1-c), \tag{4.154}$$

$$\sigma_B^2 = \frac{1+\phi\,\sigma^2[\infty]}{2(1+\chi)^2} + \frac{1}{2}(1-\phi\,\sigma^2[\infty]). \tag{4.155}$$

We see that equation (4.154) is now identical to equation (3.68), and that the difference between the two expressions (4.154) and (4.155) is only in the substitution $\phi\,\sigma^2[\infty] \to c$.

The results of testing the two volatility approximations (4.154) and (4.155) against numerical simulations are shown in Figs 4.4 and 4.5. These figures reveal that approximation (4.155) is the more accurate one in the absence of decision noise, whereas with decision noise the two approximations are either virtually indistinguishable (multiplicative noise) or of a similar quality (additive noise). We also note that, in contrast to the observables c and ϕ (and in contrast to the approximated formulas (4.154) and (4.155)), the volatility does show a dependence on the strength of additive noise for $\alpha > \alpha_c$. In terms of the market interpretation of MG, the introduction of additive decision noise is seen to have the beneficial effect of removing the high volatility solutions following tabula rasa initialization in the low α regime (unlike multiplicative decision noise). The explanation is that the noise disrupts the oscillations responsible for these high volatility solutions. As mentioned earlier, in order to avoid spurious results in

[20] Alternatively, one could say that we use equation (4.153) to determine the unknown quantity V in equation (4.151) (or vice versa).

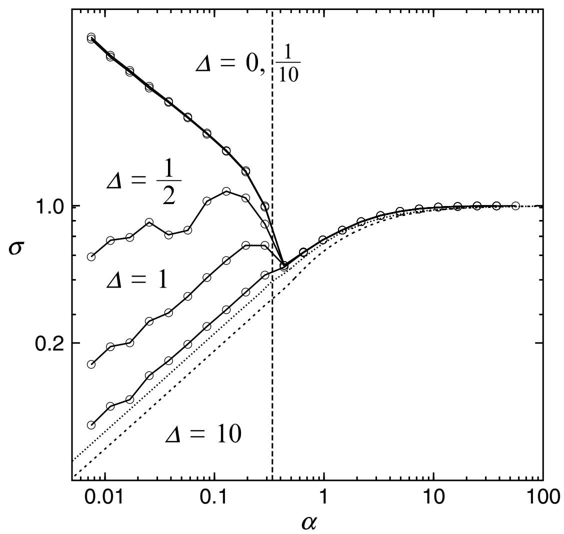

Fig. 4.4 The approximations for the volatility σ in terms of persistent order parameters only, together with simulation data (connected markers), for the fake memory batch MG without decision noise: viz. formula (4.154), derived earlier in the chapter on replica theory (short dashes), and formula (4.155) (dotted line, slightly higher). Initial conditions: $|q_i(0)| = \Delta$ for all i, with $\Delta \in \{0,\ 0.1,\ 0.5,\ 1,\ 10\}$ (from top to bottom). The vertical dashed line marks the predicted critical value $\alpha_c \approx 0.33740$.

numerical experimentation one has to be aware that in the case of additive decision noise the relaxation times are significantly larger than those of the noise-free MG.

In view of the assumptions that were found to be necessary to eliminate non-persistent observables from our expression for the volatility, we would seem obliged to at least clarify the implications of finding $V = 1$. To do so we introduce the Fourier transform $\hat{G}(\omega) = \sum_{t\geq 0} G(t)\mathrm{e}^{-\mathrm{i}\omega t}$, with $\omega \in [-\pi, \pi]$, which allows us to write

$$V = \int_{-\pi}^{\pi} \frac{\mathrm{d}\omega}{2\pi} |1 + \hat{G}(\omega)|^{-2}.$$

If, for instance, we would approximate $G(t)$ by an exponential function, the constraint $\chi = \sum_t G(t)$ would force us to choose the form $G(t) = \chi(\mathrm{e}^z - 1)\mathrm{e}^{-zt}$ with $z > 0$. For such functions one would have $\hat{G}(\omega) = \chi(1 - \mathrm{e}^{-z})/(\mathrm{e}^{\mathrm{i}\omega} - \mathrm{e}^{-z})$ and

$$V = \int_{-\pi}^{\pi} \frac{\mathrm{d}\omega}{2\pi}\, \frac{|\mathrm{e}^{\mathrm{i}\omega} - \mathrm{e}^{-z}|^2}{|\mathrm{e}^{\mathrm{i}\omega} - \mathrm{e}^{-z} + \chi(1 - \mathrm{e}^{-z})|^2}. \tag{4.156}$$

We conclude that, in combination with the constraint $\sum_t G(t) = \chi$, the statement $V = 1$ implies that the relaxation of the response function must be very slow (i.e. z is close to zero).

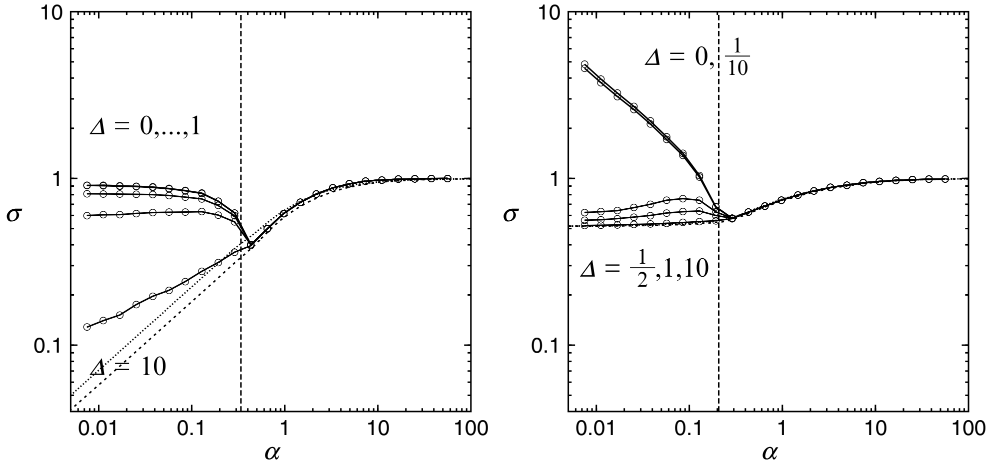

Fig. 4.5 The approximations for the volatility σ in terms of persistent order parameters only, together with simulation data (connected markers), for the fake memory batch MG with decision noise: viz. formula (4.154), derived earlier in the chapter on replica theory (short dashes), and formula (4.155) (dotted line, slightly higher). Left: additive noise, of strength $T = 1$. Right: multiplicative noise, of strength $T = 1$. Initial conditions: $|q_i(0)| = \Delta$ for all i, with $\Delta \in \{0,\ 0.1,\ 0.5,\ 1,\ 10\}$ (from top to bottom). The vertical dashed lines mark the critical values $\alpha_c \approx 0.33740$ (left), and $\alpha_c(1) \approx 0.20693$.

4.7 Stationary state in the non-ergodic regime

In the non-ergodic regime $\alpha < \alpha_c$ we know that exact results will be very difficult to obtain, but at least we have an exact starting point, namely the effective single-trader equations (4.55). We also know from our analysis of the solution for short times that, dependent upon the realization of the effective noise, the system has the tendency to develop asymptotic solutions which are either fixed-points, or period-two oscillations, or randomly perturbed period-two oscillations. In this section we show how such special solutions can be constructed, via suitable ansätze, restricting ourselves to the case of absent decision noise.

It should be made clear at the outset that the level of rigour of the procedures to be explained below is lower than that which we have by now become accustomed to. Solving the effective single-agent dynamics in the non-ergodic regime is without doubt the one stage of the generating functional analysis research programme where much more work is needed. Taking a more optimistic view, one could equally argue that it is precisely the high level of exactness of the generating functional method and its broader range of applicability that makes this particular stage of the programme stand out.

4.7.1 Approximate stationary solutions: the ansatz

In the absence of decision noise we may put $\sigma[q, z] = \text{sgn}[q]$, and our effective single-agent process (4.55) simplifies to

$$q(t+1) = q(t) + \theta(t) - \alpha \sum_{t' \leq t} (\mathbf{1} + G)^{-1}_{tt'} \text{sgn}[q(t')] + \sqrt{\alpha}\, \eta(t). \quad (4.157)$$

We restrict ourselves again to TTI solutions, but now (in view of the very definition of the transition to the non-ergodic regime) with anomalous response, i.e.

$$C_{tt'} = C(t - t'), \quad G_{tt'} = G(t - t'), \quad \chi = \sum_t G(t) = \infty. \quad (4.158)$$

It follows from $\chi \to \infty$ that now also $\sum_t (\mathbf{1} + G)^{-1}(t) = (1 + \chi)^{-1} \to 0$. The role of the static susceptibility χ will here, it turns out, to some extent be taken over by the susceptibility $\hat{\chi}$ of the system to persistent *oscillatory* perturbations

$$\hat{\chi} = \sum_t G(t)(-1)^t. \quad (4.159)$$

A simple adaptation of our earlier proof (4.117) of $\sum_t (\mathbf{1} + G)^{-1}(t) = (1 + \chi)^{-1}$ shows that now we must have $\sum_t (\mathbf{1} + G)^{-1}(t)(-1)^t = (1 + \hat{\chi})^{-1}$. In view of the special roles of both $\hat{\chi}$ and χ we restrict ourselves to perturbation fields of the form $\theta(t) = \theta_1 + \theta_2(-1)^t$, so that we may compute our two susceptibilities from

$$\chi = \frac{\partial}{\partial \theta_1} \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{t \leq \tau} \langle \text{sgn}[q(t)] \rangle_\star, \quad \hat{\chi} = \frac{\partial}{\partial \theta_2} \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{t \leq \tau} (-1)^t \langle \text{sgn}[q(t)] \rangle_\star. \quad (4.160)$$

We will make an ansatz which is equivalent to the claim that our system consists only of agents which are either frozen, or oscillating with period two, i.e.

$$C(t) = \phi + (1 - \phi)(-1)^t. \quad (4.161)$$

This enables us to calculate the covariances of the effective noise $\eta(t)$ in equation (4.157):

$$\begin{aligned} \Sigma(t - t') &= \sum_{ss'} (\mathbf{1} + G)^{-1}(t - s) \Big[1 + \phi + (1 - \phi)(-1)^{s+s'}\Big] (\mathbf{1} + G)^{-1}(t' - s') \\ &= \frac{(1 - \phi)(-1)^{t-t'}}{(1 + \hat{\chi})^2}. \end{aligned} \quad (4.162)$$

We then find, in turn, that we may write our effective noise variables as

$$\eta(t) = \frac{z\sqrt{1 - \phi}}{1 + \hat{\chi}} (-1)^t \quad (4.163)$$

with z denoting a zero average and unit variance frozen Gaussian variable. In fact z is the only remaining source of randomness in the process (4.157), apart from initial conditions, and thus plays a role similar to that of the persistent noise $\overline{\eta}$ in our analysis of the stationary state in the ergodic regime. Insertion of equation (4.163) into equation (4.157) gives in the stationary state

$$\begin{aligned} q(t+1) = q(t) + \theta_1 + (-1)^t \Big\{\theta_2 + \frac{z\sqrt{\alpha(1-\phi)}}{1+\hat{\chi}}\Big\}, \\ -\alpha \sum_{t' \le t} (\mathbf{1}+G)^{-1}(t-t')\text{sgn}[q(t')]. \end{aligned} \tag{4.164}$$

Strictly frozen and strictly oscillating (fickle) solutions must at all times obey $\text{sgn}[q(t)] = S_{\text{fr}}$ versus $\text{sgn}[q(t)] = S_{\text{fi}}(-1)^t$, respectively, with $S_{\text{fr}}, S_{\text{fi}} \in \{-1, 1\}$. According to equation (4.164) we must therefore have

$$\text{frozen}: \quad q(t+1) = q(t) + \theta_1 + (-1)^t \Big\{\theta_2 + \frac{z\sqrt{\alpha(1-\phi)}}{1+\hat{\chi}}\Big\}, \tag{4.165}$$

$$\text{fickle}: \quad q(t+1) = q(t) + \theta_1 + (-1)^t \Big\{\theta_2 + \frac{z\sqrt{\alpha(1-\phi)}}{1+\hat{\chi}} - \frac{\alpha S_{\text{fi}}}{1+\hat{\chi}}\Big\}. \tag{4.166}$$

Solution of these two equations for $q(t)$, followed by determining the associated conditions for existence, based on the required behaviour of $\text{sgn}[q(t)]$, gives

$$\text{frozen}: \quad q(t) = \overline{q} + \theta_1 t + \frac{1}{2}(-1)^t \Big\{\theta_2 + \frac{z\sqrt{\alpha(1-\phi)}}{1+\hat{\chi}}\Big\}, \tag{4.167}$$

$$S_{\text{fr}} = \text{sgn}[\overline{q} + \theta_1 t], \tag{4.168}$$

$$\text{condition}: \quad |\overline{q} + \theta_1 t| > \frac{1}{2}\Big|\theta_2 + \frac{z\sqrt{\alpha(1-\phi)}}{1+\hat{\chi}}\Big|, \tag{4.169}$$

$$\text{fickle}: \quad q(t) = \overline{q} + \theta_1 t + \frac{1}{2}(-1)^t \Big\{\theta_2 + \frac{z\sqrt{\alpha(1-\phi)}}{1+\hat{\chi}} - \frac{\alpha S_{\text{fi}}}{1+\hat{\chi}}\Big\}, \tag{4.170}$$

$$S_{\text{fi}} = -\text{sgn}\Big[\theta_2 + \frac{z\sqrt{\alpha(1-\phi)}}{1+\hat{\chi}} - \frac{\alpha S_{\text{fi}}}{1+\hat{\chi}}\Big], \tag{4.171}$$

$$\text{condition}: \quad |\overline{q} + \theta_1 t| < \frac{1}{2}\Big|\theta_2 + \frac{z\sqrt{\alpha(1-\phi)}}{1+\hat{\chi}} - \frac{\alpha S_{\text{fi}}}{1+\hat{\chi}}\Big|. \tag{4.172}$$

It should be emphasized that the long-time averages $\overline{q}$ in these expressions do not represent only the initial conditions, but rather the cumulative effect of initial conditions and the (stochastic) transients of the dynamical process. Finally we may also extract from equation (4.162) an expression for the volatility σ:

$$\sigma = \frac{\sqrt{1-\phi}}{\sqrt{2}|1+\hat{\chi}|}. \tag{4.173}$$

4.7.2 Self-consistency and simplification

It is clear from equation (4.172) that, due to $t \to \infty$, the average $\langle \text{sgn}[q(t)]\rangle_\star$ will depend discontinuously on whether $\theta_1 = 0$ or $\theta_1 \neq 0$, so that via equation (4.160) we confirm that indeed $\chi = \infty$ at $\theta_1 = 0$. We have thereby established the self-consistency of our ansatz. The fields θ_1 are now no longer needed and can be put to zero.

We may also work out the second identity in equation (4.160), using the fact that derivation with respect to the field θ_2 can be replaced (modulo constants) by derivation with respect to the Gaussian variable z (upon which the field θ_2 can be put to zero and the z-derivative be dealt with via integration by parts)

$$\hat{\chi} = \lim_{\theta_2 \to 0} \frac{\partial}{\partial \theta_2} \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{t \leq \tau} (-1)^t \left[\phi \langle S_{\text{fr}}\rangle_{\text{frozen}} + (1-\phi)\langle S_{\text{fi}}\rangle_{\text{fickle}}(-1)^t\right]$$
$$= \frac{1+\hat{\chi}}{\sqrt{\alpha(1-\phi)}} \left[(-1)^t \phi \langle z \;\text{sgn}[\overline{q}]\rangle_{\text{frozen}} + (1-\phi)\langle z S_{\text{fi}}\rangle_{\text{fickle}}\right]. \tag{4.174}$$

Since the left-hand side of this identity must be independent of t, we conclude that for $\alpha \to 0$ the following two statements must both be true

$$\frac{1}{1+\hat{\chi}} = 1 - \sqrt{\frac{1-\phi}{\alpha}}\; \langle z S_{\text{fi}}\rangle_{\text{fickle}}, \quad \langle z \;\text{sgn}[\overline{q}]\rangle_{\text{frozen}} = 0. \tag{4.175}$$

Both perturbation fields $\{\theta_1, \theta_2\}$ have now vanished from the scene, and our equations can be simplified considerably. In order to calculate order parameters we no longer need the time dependence of $q(t)$ explicitly, it suffices to know $\{S_{\text{fr}}, S_{\text{fi}}\}$ and the conditions for existence of our solutions. Our expressions can be simplified further by eliminating $\hat{\chi}$ at this stage, in favour of the volatility, via equation (4.173). Upon putting $S = \text{sgn}[1+\hat{\chi}]$ and $S_{\text{fickle}} = S\text{sgn}[z]\hat{S}(|z|)$, with $S, \hat{S}(|z|) = \pm 1$, the result can be written as

$$\text{frozen}: \quad S_{\text{fr}} = \text{sgn}[\overline{q}], \tag{4.176}$$
$$\text{condition}: \quad |\overline{q}| > \frac{1}{2}\sigma\sqrt{2\alpha}\,|z|, \tag{4.177}$$
$$\text{fickle}: \quad S_{\text{fi}} = S\text{sgn}[z]\hat{S}(|z|), \tag{4.178}$$
$$\text{condition}: \quad |\overline{q}| < \frac{1}{2}\sigma\sqrt{2\alpha}\left||z| - \frac{S\sqrt{\alpha}}{\sqrt{1-\phi}}\hat{S}(|z|)\right|, \tag{4.179}$$

whereas the first equation of (4.175) and expression (4.171) translate into the following three conditions

$$\sigma = \frac{\sqrt{1-\phi}}{\sqrt{2}}\left|1 - \sqrt{\frac{1-\phi}{\alpha}}\; S\langle |z|\hat{S}(|z|)\rangle_{\text{fickle}}\right|, \tag{4.180}$$

$$S = \text{sgn}\Big[1 - \sqrt{\frac{1-\phi}{\alpha}}\, S\langle |z|\hat{S}(|z|)\rangle_{\text{fickle}}\Big], \tag{4.181}$$

$$\hat{S}(|z|) = \text{sgn}\Big[S\hat{S}(|z|) - |z|\sqrt{\frac{1-\phi}{\alpha}}\Big]. \tag{4.182}$$

Numerical simulations show that the correlation function takes the assumed form (4.161) strictly speaking only for $\alpha \to 0$. For finite α, the assumed strict division into purely frozen and purely oscillating solutions holds only approximately. Here small fluctuations will occasionally succeed in inducing frozen agents to oscillate and vice versa, giving rise to intermediate solutions. For this reason we now turn to the limit $\alpha \to 0$.

4.7.3 The limit $\alpha \to 0$: high volatility states

It turns out that for $\alpha \to 0$ there are two types of solutions to be distinguished. This follows from the structure of our equations, but can also be observed very clearly when measuring the ratio $(1-\phi)/\alpha$ in simulations, as shown in Fig. 4.6. The first type of

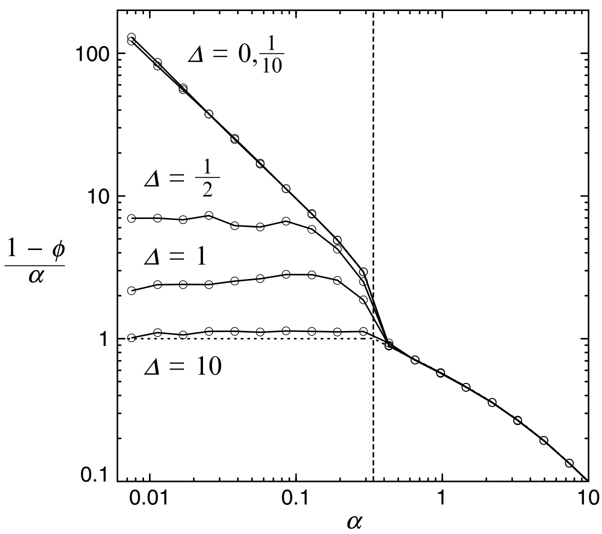

Fig. 4.6 Here we show numerical simulation data on the ratio $(1-\phi)/\alpha$ in the batch MG without decision noise, together with the theoretical prediction derived for $\alpha > \alpha_c$ (dashed curve) and its continuation in the regime $\alpha < \alpha_c$. The simulation data correspond to the connected markers. Initial conditions: $|q_i(0)| = \Delta$ for all i, with $\Delta \in \{0,\ 0.1,\ 0.5,\ 1,\ 10\}$ (from top to bottom). The vertical dashed line marks the critical value $\alpha_c \approx 0.33740$. Clearly, for $\Delta = 0, \frac{1}{10}$ we have $\lim_{\alpha\to 0}\phi = \phi_0 < 1$ and hence $(1-\phi)/\alpha = \mathcal{O}(\alpha^{-1})$ for $\alpha \to 0$, whereas for $\Delta = \frac{1}{2}, 1, 10$ we have $\lim_{\alpha\to 0}(1-\phi)/\alpha = \kappa_\Delta > 0$.

solution corresponds to $\lim_{\alpha\to 0}\phi = \phi_0 < 1$, and the second to $\lim_{\alpha\to 0}\phi_0 = 1$; our previous numerical simulations allow us to identify these solutions as the familiar high volatility and low volatility states, respectively. In fact Fig. 4.6 suggests further that the low volatility solutions scale more specifically as $\phi = 1 - \kappa_{\Delta}\alpha + \cdots$ for $\alpha \to 0$, with κ_{Δ} decreasing monotonically with the bias $\Delta = |q_0|$ in the initial relative strategy valuations.

Let us first deal with $\phi_0 < 1$. Here the two existence conditions (4.177) and (4.179) become complementary in leading order in α, and $\lim_{\alpha\to 0}\hat{S}(|z|) = -1$. We are just left with equations (4.180) and (4.181), which take the simpler form

$$\sigma = \frac{\sqrt{1-\phi}}{\sqrt{2}}\left|1 + \sqrt{\frac{1-\phi}{\alpha}}\, S\langle |z|\rangle_{\text{fickle}}\right|, \tag{4.183}$$

$$S = \text{sgn}\left[1 + \sqrt{\frac{1-\phi}{\alpha}}\, S\langle |z|\rangle_{\text{fickle}}\right]. \tag{4.184}$$

To obtain a fully closed set we finally need to work out expressions for ϕ and for $\langle |z|\rangle_{\text{fickle}}$, a task which has by now become relatively easy

$$\phi_0 = \int Dz \int \mathrm{d}\overline{q}\; P(\overline{q}|z)\; \theta\left[|\overline{q}| - \frac{1}{2}\sigma\sqrt{2\alpha}|z|\right], \tag{4.185}$$

$$\langle |z|\rangle_{\text{fickle}} = \frac{1}{1-\phi_0}\int Dz\; |z| \int \mathrm{d}\overline{q}\; P(\overline{q}|z)\; \theta\left[\frac{1}{2}\sigma\sqrt{2\alpha}|z| - |\overline{q}|\right]. \tag{4.186}$$

It immediately follows from equation (4.185) that $\sigma = \mathcal{O}(\alpha^{-\frac{1}{2}})$, since otherwise we would run into the contradiction $\phi_0 = 1$ (after all, we assumed $\phi_0 < 1$). This explains the observed power law of the high volatility solution, for small α (see the data in Fig. 4.4).

To say more, we will have to make further simplifying assumptions. These will, however, lead to remarkably accurate predictions. We put $\sigma = \tilde{\sigma}/\sqrt{\alpha} + \mathcal{O}(\alpha^0)$, and we will assume that the effective initial conditions $\overline{q}$ are statistically independent of the frozen noise z, i.e. $P(q|z) = P(q)$. This allows us to integrate out the Gaussian variable z, and replace equations (4.185) and (4.186) in the limit $\alpha \to 0$ by

$$\begin{aligned}\phi_0 &= \int \mathrm{d}\overline{q}\; P(\overline{q}) \int Dz\; \theta\left[\frac{2|\overline{q}|}{\tilde{\sigma}\sqrt{2}} - |z|\right] \\ &= \int \mathrm{d}\overline{q}\; P(\overline{q})\; \text{Erf}[|\overline{q}|/\tilde{\sigma}], \end{aligned} \tag{4.187}$$

$$\langle |z|\rangle_{\text{fickle}} = \frac{1}{1-\phi_0}\int \mathrm{d}\overline{q}\; P(\overline{q}) \int Dz\; |z|\theta\Big[\frac{1}{2}\tilde{\sigma}\sqrt{2}|z| - |\overline{q}|\Big]$$

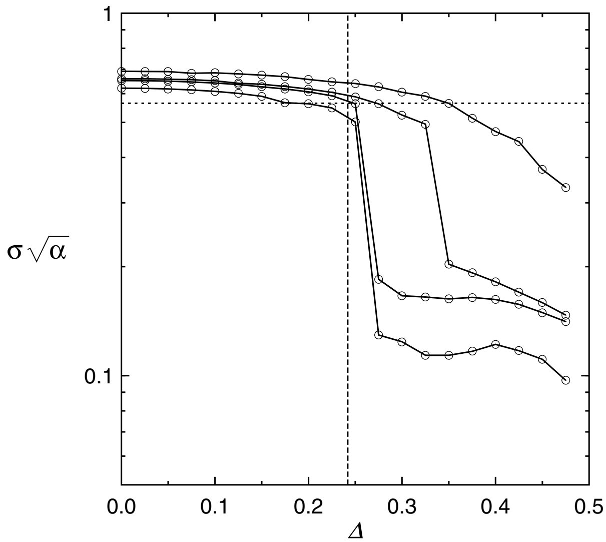

Fig. 4.7 The re-scaled volatility $\tilde{\sigma} = \sigma\sqrt{\alpha}$ as observed in numerical simulations (connected markers), as functions of the degree of initial bias $\Delta = |q_0|$, for the fake memory batch MG without decision noise. Data are shown for $\alpha \in \{0.01, 0.02, 0.04, 0.08\}$, from bottom to top in the right part of the picture. The vertical dashed line marks the predicted critical value $q_c = 1/\sqrt{2\pi e} \approx 0.242$. The horizontal dotted line marks the predicted upper limit $\tilde{\sigma}_{\max} = 1/\sqrt{\pi} \approx 0.5642$ for $\alpha \to 0$.

$$= \frac{1}{1-\phi_0}\sqrt{\frac{2}{\pi}}\int \mathrm{d}\overline{q}\; P(\overline{q})\mathrm{e}^{-\overline{q}^2/\tilde{\sigma}}. \tag{4.188}$$

Insertion of $\langle |z| \rangle_{\text{fickle}}$ into the condition (4.181) shows that the latter reduces to the trivially correct statement $S = S$. Now all of our existence conditions are satisfied, and insertion of $\langle |z| \rangle_{\text{fickle}}$ into (4.183) leads us to a very compact description, with a closed equation for $\tilde{\sigma} = \lim_{\alpha \to 0} \sigma\sqrt{\alpha}$, and a second equation expressing $\phi_0 = \lim_{\alpha \to 0} \phi$ in terms of $\tilde{\sigma}$:

$$\tilde{\sigma} = \frac{1}{\sqrt{\pi}}\int \mathrm{d}\overline{q}\; P(\overline{q})\; \mathrm{e}^{-\overline{q}^2/\tilde{\sigma}^2}, \quad \phi_0 = \int \mathrm{d}\overline{q}\; P(\overline{q})\; \mathrm{Erf}\left[\frac{|\overline{q}|}{\tilde{\sigma}}\right]. \tag{4.189}$$

Although $\tilde{\sigma} = 0$ is always a solution of the first equation, it would give $\phi_0 = 1$ in the second equation, and is therefore to be rejected. We see that any non-trivial solution solution $\tilde{\sigma}$ would have to be bounded by $\tilde{\sigma} < \pi^{-\frac{1}{2}}$, i.e. $\sigma < (\pi\alpha)^{-\frac{1}{2}}$ (which is found to be a surprisingly accurate claim, when compared to e.g. our earlier volatility simulations, or to the data in Fig. 4.7).

Finally we may put to use our (exact) solution (4.85) for finite times and small α, which shows that at least for times which do not scale with α^{-1} the time average of $q(t)$

(which will asymptotically reduce to our present $\overline{q}$) is the sum of the initial condition q_0 and a zero-average Gaussian variable, whose width Λ is a highly complicated object reflecting the transients of the dynamics. Thus we may with reason approximate $P(\overline{q}) = (\Lambda\sqrt{2\pi})^{-1}\mathrm{e}^{-\frac{1}{2}[\overline{q}-q_0]^2/\Lambda^2}$, and find our equation for non-zero solutions $\tilde{\sigma}$ in equation (4.189) simplifying further to

$$\tilde{\sigma}^2 + 2\Lambda^2 = \pi^{-1}\mathrm{e}^{-2q_0^2/(\tilde{\sigma}^2+2\Lambda^2)}. \tag{4.190}$$

We conclude that $\tilde{\sigma} = \lim_{\alpha\to 0}\sigma\sqrt{\alpha}$ can be written in terms of the solution y of a transcendental equation:

$$\tilde{\sigma} = \left[\frac{2q^2(0)}{y} - 2\Lambda^2\right]^{\frac{1}{2}}, \quad q^2(0) = \frac{y}{2\pi}\mathrm{e}^{-y}. \tag{4.191}$$

For $q(0) \to 0$ we find $\sigma = (\alpha\pi)^{-\frac{1}{2}}\sqrt{1-2\pi\Lambda^2}$, hence we must obviously require $\Lambda^2 < 1/2\pi$. Since we cannot calculate or estimate the width Λ of the effective Gaussian noise term without solving our dynamic order parameter equations more extensively, it is quite satisfactory that several interesting properties of the solution are found to be independent of Λ. For instance, the present solution predicts the existence of a critical value q_c such that for $|q(0)| > q_c$ the high volatility state cannot exist:

$$q_c = \max_{y\geq 0}\left[\frac{y}{2\pi}\mathrm{e}^{-y}\right]^{\frac{1}{2}} = (2\pi e)^{-\frac{1}{2}} \approx 0.242. \tag{4.192}$$

Numerical simulations appear to support both the existence and the predicted order of magnitude of a critical value $q_c = (2\pi e)^{-\frac{1}{2}} \approx 0.242$ surprisingly well, see Fig. 4.7. Furthermore, also the predicted upper bound $\tilde{\sigma} \leq \pi^{-\frac{1}{2}} \approx 0.5642$ is not far off.[21] The observed violations of the upper bound suggest that either the assumed independence of the variables $\overline{q}$ and z is violated, or that finite size effects in our simulations are non-negligible[22] (or both).

[21] It should be emphasized that the prediction in equation (4.192) for the maximum value q_c of $|q_0|$ to allow for a high volatility state is in fact an upper bound, as this value can be achieved only for $\Lambda \downarrow 0$ whereas one should obviously expect $\Lambda > 0$.

[22] Finite size effects unfortunately also prevent us from investigating yet smaller values of α numerically. For instance, in the simulations of Fig. 4.7 we have used $N = 4000$. Hence $\alpha = 0.01$ (the smallest value of α for which a curve shown) already corresponds to only $p = 40$, whereas our theory relies on p diverging.

Our simple ansatz for the high volatility solution has thereby allowed us to explain (i) the scaling with α and the magnitude of the prefactor (in terms of a tight bound) of the volatility, as well as (ii) its dependence on the initial conditions q_0, both qualitatively and quantitatively. Given the approximate nature of the various arguments, one might call this a surprising harvest.

4.7.4 The limit $\alpha \to 0$: low volatility states

We note that in the above analysis the condition $\lim_{\alpha\to 0}\phi < 1$ can be weakened to $\lim_{\alpha\to 0}\alpha/(1-\phi) = 0$. The above high volatility solution ceases to hold, however, at the point where the fraction ϕ of frozen agents scales as $\phi = 1-\kappa\alpha+\mathcal{O}(\alpha^2)$. In this latter regime we can immediately verify that our equations (4.180)–(4.182) admit a solution with $S=1$ and $\hat{S}(|z|)=-1$, and

$$\sigma = \sqrt{\frac{\alpha\kappa}{2}}\left[1+\sqrt{\kappa}\,\langle|z|\rangle_{\text{fickle}}\right]+\mathcal{O}(\alpha). \tag{4.193}$$

The classification conditions (4.177) and (4.179) are now no longer complementary. leaving a regime where both frozen and fickle solutions are possible

$$\begin{aligned}
\text{frozen}: &\quad S_{\text{fr}} = \text{sgn}[\overline{q}]. && (4.194)\\
\text{condition}: &\quad |\overline{q}| > \frac{1}{2}\sqrt{2\alpha}\,|z| + \mathcal{O}(\alpha^{\frac{3}{2}}), && (4.195)\\
\text{fickle}: &\quad S_{\text{fi}} = S\text{sgn}[z]\hat{S}(|z|), && (4.196)\\
\text{condition}: &\quad |\overline{q}| < \frac{1}{2}\sqrt{2\alpha}\,|z| + \frac{\sqrt{\alpha}}{\sqrt{2\kappa}} + \mathcal{O}(\alpha^{\frac{3}{2}}). && (4.197)
\end{aligned}$$

For $\alpha \to 0$ all solutions are frozen, which is consistent with our starting point $\phi = 1-\kappa\alpha+\cdots$ To determine κ we would have to resolve the classification problem posed by equations (4.195) and (4.197) in the regime $|\overline{q}| \in [\frac{1}{2}\sqrt{2\alpha}\,|z|, \frac{1}{2}\sqrt{2\alpha}\,|z|+\frac{\sqrt{\alpha}}{\sqrt{2\kappa}}]$, which requires more detailed knowledge of the transients of the dynamics. Furthermore, even if one were to reduce one's ambition and use equations (4.195) and (4.197) for establishing upper and lower bounds for κ, it would soon become apparent that here the previously successful independence assumption $P(\overline{q}|z) = P(\overline{q})$ followed by $P(\overline{q}) = (2\pi\Lambda^2)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}(\overline{q}-q_0)^2/\Lambda^2}$ (with q_0 and Λ independent of α) would fail to give a finite nonzero value for κ. There is a limit to what can be achieved with crude approximations. In a subsequent Chapter we will find that the difficulties in constructing stationary state solutions for $\alpha < \alpha_c$ go deeper than just the analytical treatment of objects such as $P(\overline{q}|z)$ in the present ansätze; we may have to question even the time-translation invariance of our order parameters.

4.8 An assessment

It should be clear that, compared with the pseudo-equilibrium replica calculations in Chapter 3, the generating functional method has increased both the scope and the level of rigour of our mathematical analysis of MG significantly. The central result is the effective single-agent equation (4.55), which represents a fully exact and closed theory for the correlation and response functions $\{C, G\}$ in the game, in the limit $N \to \infty$. From this description one cannot only obtain as a spin-off all those results of the replica method that were correct (with these results now being derived in a rigorous manner), but one has in addition a fully exact description of the MG's dynamics, including all transient phenomena and including the dynamics in the notorious non-ergodic regime. Furthermore, we now even have an exact expression for the volatility in terms of macroscopic (dynamic) order parameters.

The question of whether one may now regard the batch MG model as solved and understood can only have subjective answers. Let us therefore instead try to summarize the state of play as follows. Whereas the main achievement of the pseudo-equilibrium replica approach was to show that MGs can indeed be analyzed quantitatively with statistical mechanical tools, the emphasis has now shifted within the statistical mechanics of MGs from finding a method with which to calculate *exact dynamical* equations for order parameters and other relevant observables, to the problem of how to *solve* these exact equations. This, in a nutshell, is the main achievement of the generating functional method.

To what extent it is possible to extract such solutions for the order parameters $\{C, G\}$ from the effective single-agent equation (4.55) depends on one's energy and persistence. Whereas it would appear that the ergodic regime $\alpha > \alpha_c$ of the batch MG no longer holds significant secrets, so far only modest advances have been made in the non-ergodic regime $\alpha < \alpha_c$. For low α it has been possible to find exact solutions for the first few time steps, and we also have been able to explain the existence and scaling with α of both the high and low volatility stationary states. However, in the latter area, the analysis of both transient and stationary state solutions in the non-ergodic phase, one would hope and expect to see further progress and reduced reliance on approximations.

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5. Dynamics of the on-line MG with fake memory

Let us now return to our earlier microscopic equations (2.22) and (2.23), which define the original $S = 2$ MG with 'fake' memory. In Chapter 3 we first approximated these equations by replacing the microscopic variables by their averages over both sources of fluctuations, i.e. over the decision noise variables $\{z_i(\ell)\}$ and over the 'fake' memory realizations $\{\mu(\ell)\}$, resulting in equations (2.42) and (2.43), and we subsequently analyzed the stationary state of the model using the replica method. In Chapter 4 we restored the decision noise fluctuations, but we again replaced the random memory selections by averaging over all such selections at each time step. We were then able to solve the resulting 'batch' dynamics (2.44), within limits, using generating functional analysis.

Having sharpened our teeth, we now try to apply the generating functional techniques to the original MG equations (2.22) and (2.23), still with 'fake' memory but now at least with *all* sources of fluctuations included. Given the considerable level of detail in Chapter 4, where the generating functional methods were introduced, we will here be able to concentrate largely on those aspects of our analysis and the dynamical solutions that differ from their batch MG counterparts. Since time is now continuous, the generating functional analysis, leading again to exact closed dynamical laws for correlation and response functions in terms of a single 'effective agent', will now involve path integrals.

5.1 Definitions

5.1.1 The 'on-line' process and its discretization

We thus return to the stochastic 'on-line' equations (2.22) and (2.23), which we will study in their temporally regularized probabilistic formulation (2.25), (2.28), and (2.34):

$$\frac{\mathrm{d}}{\mathrm{d}t}p_t(\mathbf{q}) = \frac{2p}{\tilde{\eta}}\Big\{\int \mathrm{d}\mathbf{q}'\, W(\mathbf{q}|\mathbf{q}')p_t(\mathbf{q}') - p_t(\mathbf{q})\Big\}, \tag{5.1}$$

$$W(\mathbf{q}|\mathbf{q}') = \frac{1}{p}\sum_{\mu=1}^{p}\int \mathrm{d}\mathbf{z}\, P(\mathbf{z})\prod_{i=1}^{N}\delta\left[q_i - q_i' + \frac{\tilde{\eta}}{\sqrt{N}}\xi_i^\mu A^\mu[\mathbf{q}',\mathbf{z}]\right]. \tag{5.2}$$

Although one might be tempted to simply choose $\tilde{\eta} = 1$ for simplicity, retaining for now this learning rate variable will allow us to make contact later with the results of Chapter 3, which one ought to recover in the limit $\tilde{\eta} \to 0$ (where all fluctuations must be averaged out automatically). We note that, upon writing the various δ-functions in integral representation, the processes (5.1) and (5.2) can also be written in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}p_t(\mathbf{q}) = \frac{2}{\tilde{\eta}}\int \mathrm{d}\mathbf{q}'\, p_t(\mathbf{q}')\int \frac{\mathrm{d}\hat{\mathbf{q}}}{(2\pi)^N}\mathrm{e}^{\mathrm{i}\sum_i \hat{q}_i[q_i - q_i']} \\ \times \sum_{\mu=1}^{p}\int \mathrm{d}\mathbf{z}\, P(\mathbf{z})\left[\mathrm{e}^{\frac{\mathrm{i}\tilde{\eta}}{\sqrt{N}}\sum_i \hat{q}_i\xi_i^\mu A^\mu[\mathbf{q}',\mathbf{z}]} - 1\right].\end{aligned} \tag{5.3}$$

Since we still have fake (i.e. random) memory, long-time averages of global bid co-variances are again automatically converted into averages over all possible memory variables $\mu \in \{1,\ldots,\alpha N\}$, and the volatility matrix for the present on-line MG is a simple adaptation of the definition (4.5) which was introduced for the batch MG. If, as in the batch MG, we accept that quantities such as $\frac{1}{p}\sum_{\nu=1}^{p}\langle A^\nu[\mathbf{q}(t'),\mathbf{z}(t')]\rangle$ must be self-averaging for $N \to \infty$, i.e. independent of the microscopic disorder realization, we may write the asymptotic disorder-averaged volatility matrix again as

$$\begin{aligned}\Xi(t,t') = \lim_{N\to\infty}\Bigg\{&\frac{1}{p}\sum_{\mu=1}^{p}\overline{\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]A^\mu[\mathbf{q}(t'),\mathbf{z}(t')]\rangle} \\ &- \left[\frac{1}{p}\sum_{\mu=1}^{p}\overline{\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]\rangle}\right]\left[\frac{1}{p}\sum_{\mu=1}^{p}\overline{\langle A^\mu[\mathbf{q}(t'),\mathbf{z}(t')]\rangle}\right]\Bigg\}\end{aligned} \tag{5.4}$$

where now $t, t' \in \mathbb{R}$, and averages $\langle \cdots \rangle$ refer to the continuous time process (5.1) and (5.2). The volatility can subsequently be written as

$$\sigma^2 = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau \mathrm{d}t\, \Xi(t,t). \tag{5.5}$$

At this stage we discretize our continuous time, i.e. we put $t \to \ell\delta$ ($\ell = 0, 1, 2, \ldots$) with intervals $0 < \delta \ll 1$ which can be sent to zero *independent* of the limit $N \to \infty$

(since the regularization procedure followed in Section 2.2 leads to a continuous-time master equation for any N). Equation (5.3) then tells us that

$$p_{t+\delta}(\mathbf{q}) = p_t(\mathbf{q}) + \frac{2\delta}{\tilde{\eta}} \int \mathrm{d}\mathbf{q}'\, p_t(\mathbf{q}') \int \frac{\mathrm{d}\hat{\mathbf{q}}}{(2\pi)^N} \mathrm{e}^{\mathrm{i}\sum_i \hat{q}_i[q_i - q_i']}$$
$$\times \sum_{\mu=1}^{p} \int \mathrm{d}\mathbf{z}\, P(\mathbf{z}) \left[\mathrm{e}^{\frac{\mathrm{i}\tilde{\eta}}{\sqrt{N}} \sum_i \hat{q}_i \xi_i^\mu A^\mu[\mathbf{q}',\mathbf{z}]} - 1 \right] + \mathcal{O}(\delta^2)$$
$$= \int \mathrm{d}\mathbf{q}'\, p_t(\mathbf{q}') \int \frac{\mathrm{d}\hat{\mathbf{q}}}{(2\pi)^N} \mathrm{e}^{\mathrm{i}\sum_i \hat{q}_i[q_i - q_i']} \int \mathrm{d}\mathbf{z}\, P(\mathbf{z})$$
$$\times \left[\frac{2\delta}{\tilde{\eta}} \sum_{\mu=1}^{p} \mathrm{e}^{\frac{\mathrm{i}\tilde{\eta}}{\sqrt{N}} \sum_i \hat{q}_i \xi_i^\mu A^\mu[\mathbf{q}',\mathbf{z}]} + \left(1 - \frac{2p\delta}{\tilde{\eta}}\right) \right] + \mathcal{O}(\delta^2) \tag{5.6}$$

It follows that, in the relevant leading order in δ, the probability density of finding a *path* of microscopic states $\{\mathbf{q}(0), \mathbf{q}(\delta), \mathbf{q}(2\delta), \ldots\}$ in the discretized process can be written as

$$\mathrm{Prob}[\mathbf{q}(0), \mathbf{q}(\delta), \mathbf{q}(2\delta), \ldots] = p_0(\mathbf{q}(0)) \prod_{t>0} \tilde{W}[\mathbf{q}(t)|\mathbf{q}(t-\delta)] \tag{5.7}$$

with

$$\tilde{W}[\mathbf{q}|\mathbf{q}'] = \int \frac{\mathrm{d}\hat{\mathbf{q}}}{(2\pi)^N} \mathrm{e}^{\mathrm{i}\sum_i \hat{q}_i[q_i - q_i']} \int \mathrm{d}\mathbf{z}\, P(\mathbf{z})$$
$$\times \left[\frac{2\delta}{\tilde{\eta}} \sum_{\mu=1}^{p} \mathrm{e}^{\frac{\mathrm{i}\tilde{\eta}}{\sqrt{N}} \sum_i \hat{q}_i \xi_i^\mu A^\mu[\mathbf{q}',\mathbf{z}]} + \left(1 - \frac{2p\delta}{\tilde{\eta}}\right) \right] \tag{5.8}$$

We will ultimately be seeking to take the continuous time limit $\delta \to 0$, and find that the theory can be developed in such a way that all discrete time summations will indeed scale according to $\delta \sum_t [\cdots] = \mathcal{O}(\delta^0)$. This confirms once more that the real-valued time t as established in Chapter 2 gives indeed the natural time unit, with a well-defined limit $N \to \infty$.

Our generating functional methods require the introduction of perturbations to our dynamical equations, in order to define response functions. Here we have to be careful. Our regularization method in Section 2.2 was based on the introduction of random durations for the MG's iteration steps, so that the state $\mathbf{q}$ whose statistics are described by $p_t(\mathbf{q})$ will have been generated by a random number ℓ of individual iterations. If we were to add our perturbations $\{\theta_i(\ell)\}$ to the equations describing such individual iterations, we would *en passant* introduce uncertainty in the real time t at which these perturbations are applied. As a result we would find expressions (5.7) and (5.8) being replaced by

$$\mathrm{Prob}[\mathbf{q}(0),\mathbf{q}(\delta),\mathbf{q}(2\delta),\ldots]=p_0(\mathbf{q}(0))\prod_{t>0}\tilde{W}_{t-\delta}[\mathbf{q}(t)|\mathbf{q}(t-\delta)], \tag{5.9}$$

$$\begin{aligned}\tilde{W}_t[\mathbf{q}|\mathbf{q}'] &= \int\frac{\mathrm{d}\hat{\mathbf{q}}}{(2\pi)^N}\mathrm{e}^{\mathrm{i}\sum_i\hat{q}_i[q_i-q_i']}\int\mathrm{d}\mathbf{z}\,P(\mathbf{z})\\ &\quad\times\left[\frac{2\delta}{\tilde{\eta}}\sum_{\mu=1}^{p}\mathrm{e}^{\mathrm{i}\sum_i\hat{q}_i[\frac{\tilde{\eta}}{\sqrt{N}}\xi_i^\mu A^\mu[\mathbf{q}',\mathbf{z}]-\theta_i(t)]}+\left(1-\frac{2p\delta}{\tilde{\eta}}\right)\right].\end{aligned} \tag{5.10}$$

In contrast, if we add perturbations to the *regularized* process, we no longer introduce uncertainty into their timing, and find instead of (5.10) a more convenient expression, with the $\{\theta_i(t)\}$ disentangled from the complicated terms

$$\begin{aligned}\tilde{W}_t[\mathbf{q}|\mathbf{q}'] &= \int\frac{\mathrm{d}\hat{\mathbf{q}}}{(2\pi)^N}\mathrm{e}^{\mathrm{i}\sum_i\hat{q}_i[q_i-\theta_i(t)-q_i']}\int\mathrm{d}\mathbf{z}\,P(\mathbf{z})\\ &\quad\times\left[\frac{2\delta}{\tilde{\eta}}\sum_{\mu=1}^{p}\mathrm{e}^{\frac{\mathrm{i}\tilde{\eta}}{\sqrt{N}}\sum_i\hat{q}_i\xi_i^\mu A^\mu[\mathbf{q}',\mathbf{z}]}+\left(1-\frac{2p\delta}{\tilde{\eta}}\right)\right].\end{aligned} \tag{5.11}$$

5.1.2 The generating functional

We may now define the generating functional for our discretized stochastic process, with the perturbations added to the regularized equations, i.e. with equation (5.11) rather than (5.10), and with $t\in\{0,\delta,2\delta,3\delta,\ldots\}$:

$$\begin{aligned}Z[\boldsymbol{\psi}] &= \langle\mathrm{e}^{\mathrm{i}\sum_{t\geq0}\sum_i\psi_i(t)\sigma[q_i(t),z_i(t)]}\rangle \\ &= \int\prod_{t\geq0}\left\{\mathrm{d}\mathbf{q}(t)\,\tilde{W}_t[\mathbf{q}(t+\delta)|\mathbf{q}(t)]\right\}\,p_0(\mathbf{q}(0))\,\mathrm{e}^{\mathrm{i}\sum_{t\geq0}\sum_i\psi_i(t)\sigma[q_i(t),z_i(t)]}.\end{aligned} \tag{5.12}$$

Derivation of the generating functional (5.12) with respect to the fields $\boldsymbol{\psi}$ generates the relevant multiple time observables of the present on-line MG, exactly as before with the batch version, and again we can move the disorder variables $\{\xi_i^\mu,\omega_i^\mu\}$ into exponents (in preparation for the disorder average) upon introducing the auxiliary variables $\{w_t^\mu,x_t^\mu\}$:

$$w_t^\mu=\frac{\sqrt{2}}{\sqrt{N}}\sum_i\hat{q}_i(t)\xi_i^\mu,\quad x_t^\mu=\frac{\sqrt{2}}{\sqrt{N}}\sum_i s_i(t)\xi_i^\mu,$$

with the short-hand $s_i(t) = \sigma[q_i(t), z_i(t)]$. Upon putting $\mathcal{D}\mathbf{q} = \prod_{it}[\mathrm{d}q_i(t)\sqrt{\delta/2\pi}]$, $\mathcal{D}\mathbf{w} = \prod_{\mu t}[\mathrm{d}w_t^\mu\sqrt{\delta/2\pi}]$ and $\mathcal{D}\mathbf{x} = \prod_{\mu t}[\mathrm{d}x_t^\mu\sqrt{\delta/2\pi}]$ (with similar definitions for $\mathcal{D}\hat{\mathbf{q}}$, $\mathcal{D}\hat{\mathbf{w}}$, and $\mathcal{D}\hat{\mathbf{x}}$), our generating functional (5.12) now becomes

$$\begin{aligned}
Z[\boldsymbol{\psi}] = &\int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\; \mathrm{e}^{\mathrm{i}\delta\sum_{t\mu}[\hat{w}_t^\mu w_t^\mu + \hat{x}_t^\mu x_t^\mu]} \\
&\times \int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}}\; p_0(\mathbf{q}(0))\; \mathrm{e}^{\mathrm{i}\delta\sum_{ti}\hat{q}_i(t)[q_i(t+\delta)-q_i(t)-\theta_i(t)]} \\
&\times \langle \mathrm{e}^{-\frac{\mathrm{i}\sqrt{2}\delta}{\sqrt{N}}\sum_{\mu i}\xi_i^\mu\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]+i\sum_{ti}\psi_i(t)s_i(t)}\rangle_{\mathbf{z}} \\
&\times \prod_{t\geq 0}\left[1 + \frac{2\delta}{\tilde{\eta}}\sum_{\mu=1}^{p}\left[\mathrm{e}^{\frac{1}{2}\mathrm{i}\tilde{\eta}w_t^\mu(\sqrt{2}\Omega_\mu + x_t^\mu)} - 1\right]\right]. \qquad (5.13)
\end{aligned}$$

The complications generated by including the fluctuations due to the random fake memory 'path' $\{\mu(t)\}$, which had been 'defined away' in the batch MG (resulting in the much simpler corresponding batch expression (4.17)), are now clearly visible in the last line of equation (5.13). It is instructive to compare equation (5.13) with the expression that we would have found in the case where the perturbations had been added to the non-regularized process (with random step durations)

$$\begin{aligned}
Z[\boldsymbol{\psi}] = &\int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\; \mathrm{e}^{\mathrm{i}\delta\sum_{t\mu}[\hat{w}_t^\mu w_t^\mu + \hat{x}_t^\mu x_t^\mu]} \\
&\times \int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}}\; p_0(\mathbf{q}(0))\; \mathrm{e}^{\mathrm{i}\delta\sum_{ti}\hat{q}_i(t)[q_i(t+\delta)-q_i(t)]} \\
&\times \langle \mathrm{e}^{-\frac{\mathrm{i}\sqrt{2}\delta}{\sqrt{N}}\sum_{\mu i}\xi_i^\mu\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]+i\sum_{ti}\psi_i(t)s_i(t)}\rangle_{\mathbf{z}} \\
&\times \prod_{t\geq 0}\left[1 + \frac{2\delta}{\tilde{\eta}}\sum_{\mu=1}^{p}\left[\mathrm{e}^{\mathrm{i}[\frac{1}{2}\tilde{\eta}w_t^\mu(\sqrt{2}\Omega_\mu + x_t^\mu)-\sum_i\hat{q}_i(t)\theta_i(t)]} - 1\right]\right]. \qquad (5.14)
\end{aligned}$$

Clearly, expression (5.13) is considerably easier to handle. We now concentrate on the last line of (5.13). We note that, as $\delta \to 0$ and using $\delta\sum_t = \mathcal{O}(\delta^0)$:

$$\begin{aligned}
\prod_{t\geq 0}\left[1+\frac{2\delta}{\tilde{\eta}}\sum_\mu\left[\mathrm{e}^{\frac{1}{2}i\tilde{\eta}w_t^\mu(\sqrt{2}\Omega_\mu+x_t^\mu)}-1\right]\right] &= \prod_{t\geq 0}\mathrm{e}^{\frac{2\delta}{\tilde{\eta}}\sum_\mu\left[\mathrm{e}^{\frac{1}{2}i\tilde{\eta}w_t^\mu(\sqrt{2}\Omega_\mu+x_t^\mu)}-1\right]+\mathcal{O}(\delta^2N^2)} \\
&= \mathrm{e}^{\frac{2\delta}{\tilde{\eta}}\sum_{t\geq 0}\sum_\mu\left[\mathrm{e}^{\frac{1}{2}i\tilde{\eta}w_t^\mu(\sqrt{2}\Omega_\mu+x_t^\mu)}-1+\mathcal{O}(\delta N)\right]}
\end{aligned}$$

Hence, if we choose $\delta \ll N^{-1}$ we obtain

$$Z[\boldsymbol{\psi}] = \int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\; \mathrm{e}^{\delta\sum_{t\mu}\left[\mathrm{i}(\hat{w}_t^\mu w_t^\mu+\hat{x}_t^\mu x_t^\mu)+\frac{2}{\tilde{\eta}}[\mathrm{e}^{\frac{1}{2}i\tilde{\eta}w_t^\mu(\sqrt{2}\Omega_\mu+x_t^\mu)}-1+\epsilon_N]\right]}$$

$$
\times \int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}}\, p_0(\mathbf{q}(0))\, \mathrm{e}^{\mathrm{i}\delta \sum_{ti} \hat{q}_i(t)[q_i(t+\delta)-q_i(t)-\theta_i(t)]}
$$

$$
\times \langle \mathrm{e}^{-\frac{\mathrm{i}\sqrt{2}\delta}{\sqrt{N}} \sum_{\mu i} \xi_i^\mu \sum_t [\hat{w}_t^\mu \hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]+i\sum_{ti}\psi_i(t)s_i(t)} \rangle_{\mathbf{z}}, \tag{5.15}
$$

where $\lim_{N\to\infty} \epsilon_N = 0$. Note that we would not have had the freedom to choose $\delta \ll N^{-1}$ if we had not carried out our regularization in an N-independent way. Comparison with the corresponding stage (4.17) also shows that, for $N \to \infty$, the net effect of going from batch to the original on-line dynamics is the replacement

$$
\mathrm{i}w_t^\mu(\sqrt{2}\Omega_\mu + x_t^\mu) \;\to\; \frac{2}{\tilde{\eta}}[\mathrm{e}^{\frac{1}{2}i\tilde{\eta}w_t^\mu(\sqrt{2}\Omega_\mu+x_t^\mu)} - 1]. \tag{5.16}
$$

The batch theory can thus be obtained from our present equations by choosing $\tilde{\eta} \to 0$ and $\delta \to 1$, which is quite reasonable.

5.2 The disorder-averaged generating functional

5.2.1 Evaluation of the disorder average

We can now carry out the disorder averages, denoted as $\overline{[\cdots]}$, upon transforming the variables $\{\xi_i^\mu, \omega_i^\mu\}$ back into our original look-up table entries, via $\xi_i^\mu = \frac{1}{2}(R_\mu^{i1} - R_\mu^{i2})$ and $\Omega_\mu = \frac{1}{2}\sum_j (R_\mu^{j1} + R_\mu^{j2})/\sqrt{N}$. These disorder averages, however, are affected by our change of dynamics since the quantity Ω_μ is no longer appearing simply as an exponential term in equation (5.15), and they are consequently similar but somewhat more involved than those in Chapter 4 (to which they reduce only for $\tilde{\eta} \to 0$). As before the disorder averaging generates the dynamic order parameters $C_{tt'} = N^{-1}\sum_i s_i(t)s_i(t')$, $K_{tt'} = N^{-1}\sum_i s_i(t)\hat{q}_i(t')$, and $L_{tt'} = N^{-1}\sum_i \hat{q}_i(t)\hat{q}_i(t')$ and their conjugates. For initial conditions of the form $p_0(\mathbf{q}) = \prod_i p_0(q_i)$ one has

$$
\overline{Z[\boldsymbol{\psi}]} = \int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\, \mathrm{e}^{\mathrm{i}\delta\sum_{t\mu}[\hat{w}_t^\mu w_t^\mu + \hat{x}_t^\mu x_t^\mu]} \int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}} \prod_i p_0(q_i(0))
$$

$$
\times\, \mathrm{e}^{\mathrm{i}\delta\sum_{ti}\hat{q}_i(t)[q_i(t+\delta)-q_i(t)-\theta_i(t)]} \Bigg\langle \mathrm{e}^{\mathrm{i}\sum_{it}\psi_i(t)s_i(t)} \tag{5.17}
$$

$$
\times \prod_\mu \overline{\left[\mathrm{e}^{\frac{2\delta}{\tilde{\eta}}\sum_t\left[\mathrm{e}^{\frac{1}{2}i\tilde{\eta}w_t^\mu(\sqrt{2}\Omega_\mu+x_t^\mu)}-1+\epsilon_N\right] - \frac{\mathrm{i}\sqrt{2}\delta}{\sqrt{N}}\sum_i \xi_i^\mu \sum_t[\hat{w}_t^\mu \hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]}\right]} \Bigg\rangle_{\mathbf{z}}.
$$

We concentrate on the last line of equation (5.17), which actually contains the disorder

$$
\begin{aligned}
\prod_\mu \overline{\left[\cdots\right]} &= \prod_\mu \Bigg\{ \int \frac{\mathrm{d}\Omega \mathrm{d}\hat{\Omega}}{2\pi}\ \mathrm{e}^{\mathrm{i}\hat{\Omega}\Omega + \frac{2\delta}{\tilde{\eta}} \sum_t [\mathrm{e}^{\frac{1}{2} i\tilde{\eta} w_t^\mu (\sqrt{2}\Omega + x_t^\mu)} - 1 + \epsilon_N]} \\
&\quad \times \overline{\prod_i \left[\mathrm{e}^{-\frac{iR_\mu^{i1}}{\sqrt{2N}} \left[\frac{\hat{\Omega}}{\sqrt{2}} + \delta \sum_t [\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)]\right] - \frac{\mathrm{i}R_\mu^{i2}}{\sqrt{2N}} \left[\frac{\hat{\Omega}}{\sqrt{2}} - \delta \sum_t [\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)]\right]} \right]} \Bigg\} \\
&= \prod_\mu \Bigg\{ \int \frac{\mathrm{d}\Omega \mathrm{d}\hat{\Omega}}{2\pi}\ \mathrm{e}^{\mathrm{i}\hat{\Omega}\Omega + \frac{2\delta}{\tilde{\eta}} \sum_t [\mathrm{e}^{\frac{1}{2} i\tilde{\eta} w_t^\mu (\sqrt{2}\Omega + x_t^\mu)} - 1 + \epsilon_N] + \mathcal{O}(N^{-1})} \\
&\quad \times \mathrm{e}^{-\frac{1}{4N} \sum_i \left[\left(\frac{\hat{\Omega}}{\sqrt{2}} + \delta \sum_t [\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)] \right)^2 + \left(\frac{\hat{\Omega}}{\sqrt{2}} - \delta \sum_t [\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)] \right)^2 \right]} \Bigg\} \\
&= \prod_\mu \Bigg\{ \int \frac{\mathrm{d}\Omega \mathrm{d}\hat{\Omega}}{2\pi}\ \mathrm{e}^{\mathrm{i}\hat{\Omega}\Omega - \frac{1}{4}\hat{\Omega}^2 + \frac{2\delta}{\tilde{\eta}} \sum_t [\mathrm{e}^{\frac{1}{2} i\tilde{\eta} w_t^\mu (\sqrt{2}\Omega + x_t^\mu)} - 1 + \epsilon_N] + \mathcal{O}(N^{-1})} \\
&\quad \times \mathrm{e}^{-\frac{1}{2N} \sum_i [\delta \sum_t [\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)]]^2} \Bigg\} \\
&= \int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\ \mathrm{e}^{\mathrm{i}N\delta^2 \sum_{tt'} [\hat{C}_{tt'} C_{tt'} + \hat{K}_{tt'} K_{tt'} + \hat{L}_{tt'} L_{tt'}]} \\
&\quad \times \mathrm{e}^{-\mathrm{i}\delta^2 \sum_i \sum_{tt'} [\hat{C}_{tt'} s_i(t) s_i(t') + \hat{K}_{tt'} s_i(t) \hat{q}_i(t') + \hat{L}_{tt'} \hat{q}_i(t) \hat{q}_i(t')]} \\
&\quad \times \prod_\mu \Bigg\{ \int \frac{\mathrm{d}\Omega}{\sqrt{\pi}}\ \mathrm{e}^{-\Omega^2 + \frac{2\delta}{\tilde{\eta}} \sum_t [\mathrm{e}^{\frac{1}{2} i\tilde{\eta} w_t^\mu (\sqrt{2}\Omega + x_t^\mu)} - 1 + \epsilon_N] + \mathcal{O}(N^{-1})} \\
&\quad \times e^{-\frac{1}{2}\delta^2 \sum_{tt'} \left[\hat{w}_t^\mu L_{tt'} \hat{w}_{t'}^\mu + 2\hat{x}_t^\mu K_{tt'} \hat{w}_{t'}^\mu + \hat{x}_t^\mu C_{tt'} \hat{x}_{t'}^\mu\right]} \Bigg\}.
\end{aligned}
\tag{5.18}
$$

Insertion of this result into equation (5.17) then gives us again an expression that is fully factorized over agents, and which for $N \to \infty$ can be evaluated in the usual manner by steepest descent (i.e. saddle-point) integration

$$
\overline{Z[\psi]} = \int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\ \mathrm{e}^{N[\Psi + \Phi + \Omega + \tilde{\epsilon}_N]},
\tag{5.19}
$$

with now

$$
\Psi = \mathrm{i}\delta^2 \sum_{tt'} [\hat{C}_{tt'} C_{tt'} + \hat{K}_{tt'} K_{tt'} + \hat{L}_{tt'} L_{tt'}],
\tag{5.20}
$$

$$
\begin{aligned}
\Phi &= \alpha \log \Bigg[\int \mathcal{D}w\mathcal{D}\hat{w}\mathcal{D}x\mathcal{D}\hat{x}\ \mathrm{e}^{\mathrm{i}\delta \sum_t [\hat{w}_t w_t + \hat{x}_t x_t]} \int Du\ \mathrm{e}^{\frac{2\delta}{\tilde{\eta}} \sum_t [\mathrm{e}^{\frac{1}{2} i\tilde{\eta} w_t [u + x_t]} - 1]} \\
&\quad \times \mathrm{e}^{-\frac{1}{2}\delta^2 \sum_{tt'} [\hat{w}_t L_{tt'} \hat{w}_{t'} + 2\hat{x}_t K_{tt'} \hat{w}_{t'} + \hat{x}_t C_{tt'} \hat{x}_{t'}]} \Bigg],
\end{aligned}
\tag{5.21}
$$

$$\Omega = \frac{1}{N}\sum_i \log\langle \int \mathcal{D}q\mathcal{D}\hat{q}\; p_0(q(0))\; \mathrm{e}^{\mathrm{i}\delta\sum_t\left[\hat{q}(t)[\frac{q(t+\delta)-q(t)}{\delta}-\delta^{-1}\theta_i(t)]+\delta^{-1}\psi_i(t)\sigma[q(t),z(t)]\right]}$$
$$\times\; \mathrm{e}^{-\mathrm{i}\delta^2\sum_{tt'}[\hat{C}_{tt'}\sigma[q(t),z(t)]\sigma[q(t'),z(t')]+\hat{K}_{tt'}\sigma[q(t),z(t)]\hat{q}(t')+\hat{L}_{tt'}\hat{q}(t)\hat{q}(t')]}\rangle_{\mathbf{z}}. \tag{5.22}$$

The term $\tilde{\epsilon}_N$ will vanish as $N \to \infty$, and is independent of the fields $\{\psi_i(t)\}$ and $\{\theta_i(t)\}$. We have also used the familiar abbreviation of the Gaussian measure $Du = (2\pi)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}u^2}\mathrm{d}u$. The average $\langle\ldots\rangle_{\mathbf{z}}$ has again been reduced to a single site one: $\langle g[z_1,z_2,\ldots]\rangle_{\mathbf{z}} = \int\prod_t[\mathrm{d}z_t P(z_t)]\; g[z_1,z_2,\ldots]$. We observe in the above expressions that all time summations are paired with a corresponding factor δ, which ensures that all three functions $\{\Psi,\Phi,\Omega\}$ will remain of order one when we take the limits $N\to\infty$ and $\delta\to 0$, as they should.

5.2.2 The saddle-point equations

In the limit $N\to\infty$ we can now evaluate (5.19) by saddle-point integration.[23] We define $G_{tt'} = -iK_{tt'}$, and assume as before that imaginary saddle-points can be dealt with by shifting integration contours in order parameter space. Upon taking derivatives with respect to the generating fields, and using the normalization $\overline{Z[\mathbf{0}]} = 1$, we then find at the physical saddle-point the usual identifications

$$C_{tt'} = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\sigma[q_i(t),z_i(t)]\sigma[q_i(t'),z_i(t')]\rangle}, \tag{5.23}$$

$$G_{tt'} = \lim_{N\to\infty}\frac{1}{N}\sum_i \frac{\partial}{\partial\theta_i(t')}\overline{\langle\sigma[q_i(t),z_i(t)]\rangle}, \tag{5.24}$$

$$L_{tt'} = 0 \tag{5.25}$$

Putting $\psi_i(t) = 0$ (as these fields are no longer needed) and $\theta_i(t) = \theta(t)$ then simplifies (5.22) to

$$\Omega = \log\int \mathcal{D}q\mathcal{D}\hat{q}\;\langle M[\{q,\hat{q},z\}]\rangle_{\mathbf{z}} \tag{5.26}$$

[23] Here we are throwing a blanket over the well-known but highly non-trivial issue of the validity of saddle-point arguments in path integrals. We basically assume that the limits $\delta\to 0$ and $N\to\infty$ can be interchanged, whereas earlier we relied upon our freedom to choose $\delta \ll N^{-1}$. In other words: we are using the saddle-point argument in an integral of which the dimension diverges with N, where it, strictly speaking, cannot be used. Physically, the present saddle-point argument is correct only if the functions $\{C,K,L\}$ and their conjugates depend sufficiently smoothly on their two time arguments. In Chapter 8 we will use a different method for solving the present type of on-line models, without the temporal regularization procedure used here, and will *en passant* recover the present results in a way that shows much more transparently which are the underlying implicit smoothness assumptions.

$$M[\{q,\hat{q},z\}] = p_0(q(0))\,\mathrm{e}^{\mathrm{i}\delta\sum_t \hat{q}(t)\left[\frac{q(t+\delta)-q(t)}{\delta}-\delta^{-1}\theta(t)-\delta\sum_{t'}\hat{K}_{t't}\sigma[q(t'),z(t')]\right]}$$
$$\times\,\mathrm{e}^{-\mathrm{i}\delta^2\sum_{tt'}[\hat{C}_{tt'}\sigma[q(t),z(t)]\sigma[q(t'),z(t')]+\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')]} \tag{5.27}$$

We define the associated measure

$$\langle g[\{q,\hat{q},z\}]\rangle_\star = \frac{\int \mathcal{D}q\mathcal{D}\hat{q}\,\langle M[\{q,\hat{q},z\}]g[\{q,\hat{q},z\}]\rangle_{\mathbf{z}}}{\int \mathcal{D}q\mathcal{D}\hat{q}\,\langle M[\{q,\hat{q},z\}]\rangle_{\mathbf{z}}}. \tag{5.28}$$

Extremization of the extensive exponent $\Psi+\Phi+\Omega$ in expression (5.19) with respect to $\{C,\hat{C},K,\hat{K},L\}$ gives us the remaining saddle-point equations, which the above definitions allow us to write in compact form as

$$C_{tt'} = \langle\, \sigma[q(t),z(t)]\;\sigma[q(t'),z(t')]\,\rangle_\star, \tag{5.29}$$

$$G_{tt'} = -i\langle \sigma[q(t),z(t)]\;\hat{q}(t')\,\rangle_\star, \tag{5.30}$$

$$\hat{C}_{tt'} = \frac{i}{\delta^2}\frac{\partial\Phi}{\partial C_{tt'}},\quad \hat{K}_{tt'} = \frac{1}{\delta^2}\frac{\partial\Phi}{\partial G_{tt'}},\quad \hat{L}_{tt'} = \frac{i}{\delta^2}\frac{\partial\Phi}{\partial L_{tt'}}. \tag{5.31}$$

5.3 Simplification of the saddle-point equations

Compared with the previously studied batch dynamics, the main changes in our equations are to be found in the function Ω (5.27), which we can anticipate to give us in due course an effective single-trader problem in the form of a stochastic differential equation, and in the function Φ (5.21) where the implications of having on-line dynamics are as yet less clear. In order to proceed we have to work out this latter term Φ further.

5.3.1 Evaluation of Φ

We first rewrite Φ slightly, in order to isolate the tricky part

$$\Phi = \alpha\log\int \mathcal{D}w\mathcal{D}x\;\mathcal{D}\hat{w}\mathcal{D}\hat{x}\int Du\;\phi[\{x,w\},u]$$
$$\times\,\mathrm{e}^{-\frac{1}{2}\delta^2\sum_{tt'}[\hat{w}_tL_{tt'}\hat{w}_{t'}+2\hat{x}_tK_{tt'}\hat{w}_{t'}+\hat{x}_tC_{tt'}\hat{x}_{t'}]+i\delta\sum_t[\hat{w}_tw_t+\hat{x}_tx_t]} \tag{5.32}$$

in which

$$\phi[\{x,w\},u] = \exp\left[\frac{2\delta}{\tilde{\eta}}\sum_t[\mathrm{e}^{\frac{1}{2}i\tilde{\eta}w_t[u+x_t]}-1]\right]$$
$$= \prod_{t\geq 0}\left\{\mathrm{e}^{-2\delta/\tilde{\eta}}\sum_{n\geq 0}\frac{1}{n!}\left(\frac{2\delta}{\tilde{\eta}}\right)^n\mathrm{e}^{\frac{1}{2}i\tilde{\eta}nw_t[u+x_t]}\right\}$$

$$= \sum_{n_0,n_1,\ldots,\geq 0} \Big[\frac{1}{n_0!}\Big(\frac{2\delta}{\tilde{\eta}}\Big)^{n_0} \mathrm{e}^{-\frac{2\delta}{\tilde{\eta}}}\Big]\Big[\frac{1}{n_1!}\Big(\frac{2\delta}{\tilde{\eta}}\Big)^{n_1} \mathrm{e}^{-\frac{2\delta}{\tilde{\eta}}}\Big]\cdots \mathrm{e}^{\frac{1}{2}i\tilde{\eta}\sum_t n_t w_t[u+x_t]}.$$

We thereby see explicitly that $\phi[\ldots]$ can be written in the form of an average over statistically independent Poissonnian-distributed integers $n_t \in \{0, 1, 2, \ldots\}$

$$\phi[\{x,w\},u] = \Big\langle \exp\Big[\frac{1}{2}i\tilde{\eta}\sum_t n_t w_t[u+x_t]\Big]\Big\rangle_{\{n\}}, \tag{5.33}$$

where

$$\langle f[n_0,n_1,\ldots]\rangle_{\{n\}} = \sum_{n_0,n_1,\ldots,\geq 0}\Big[\prod_s P_{2\delta/\tilde{\eta}}[n_s]\Big]\, f[n_0,n_1,\ldots] \tag{5.34}$$

with the Poisson distribution $P_a[\ell] = (\ell!)^{-1}\mathrm{e}^{-a}a^{\ell}$. We note that the first two moments of this distribution are given by $\langle n_t\rangle = 2\delta/\tilde{\eta}$ and $\langle n_t^2\rangle = (2\delta/\tilde{\eta})^2 + 2\delta/\tilde{\eta}$.

The Poissonnian representation (5.33) allows us to proceed with the integrations over u, $\{x\}$, $\{\hat{x}\}$, and $\{w\}$ (preferably to be carried out in precisely that order), upon inserting equation (5.33) into equation (5.32). The various (standard) Gaussian integrals can be found in Appendix B. It will be helpful in the manipulations below to define the diagonal random matrix E with entries $E_{tt'} = \frac{1}{2}\tilde{\eta}n_t\delta_{tt'} = \delta.\delta_{tt'}n_t/\langle n\rangle$, as well as the previously encountered matrix D with entries $D_{tt'} = 1 + C_{tt'}$:

$$\begin{aligned}
\Phi &= \alpha\log\Big\langle \int \mathcal{D}w\mathcal{D}x\,\mathcal{D}\hat{w}\mathcal{D}\hat{x}\int \mathcal{D}u\;\mathrm{e}^{\frac{1}{2}i\tilde{\eta}\sum_t n_t w_t[u+x_t]} \\
&\quad\times \mathrm{e}^{-\frac{1}{2}\delta^2\sum_{tt'}[\hat{w}_t L_{tt'}\hat{w}_{t'} + 2\hat{x}_t K_{tt'}\hat{w}_{t'} + \hat{x}_t C_{tt'}\hat{x}_{t'}] + \mathrm{i}\delta\sum_t[\hat{w}_t w_t + \hat{x}_t x_t]}\Big\rangle_{\{n\}} \\
&= \alpha\log\Big\langle \int \mathcal{D}\hat{w}\mathcal{D}w\,\mathcal{D}\hat{x}\;\mathrm{e}^{-\frac{1}{2}\delta^2\sum_{tt'}\left[\hat{w}_t L_{tt'}\hat{w}_{t'} + 2\hat{x}_t K_{tt'}\hat{w}_{t'} + \hat{x}_t C_{tt'}\hat{x}_{t'} + w_t(\frac{n_t n_{t'}}{\langle n\rangle^2})w_{t'}\right]} \\
&\quad\times \mathrm{e}^{\mathrm{i}\delta\sum_t \hat{w}_t w_t}\int\prod_t\Big[\frac{\mathrm{d}x_t}{\sqrt{2\pi/\delta}}\Big]\;\mathrm{e}^{\mathrm{i}\delta\sum_t x_t[\hat{x}_t + w_t n_t/\langle n\rangle]}\Big\rangle_{\{n\}} \\
&= \alpha\log\Big\langle \int \mathcal{D}\hat{w}\mathcal{D}w\;\mathrm{e}^{-\frac{1}{2}\delta^2\sum_{tt'}\left[\hat{w}_t L_{tt'}\hat{w}_{t'} + w_t(\frac{n_t n_{t'}}{\langle n\rangle^2})w_{t'}\right] + \mathrm{i}\delta\sum_t \hat{w}_t w_t} \\
&\quad\times \int\prod_t\Big[\mathrm{d}\hat{x}_t\delta\Big[\hat{x}_t + \frac{w_t n_t}{\langle n\rangle}\Big]\Big]\mathrm{e}^{-\frac{1}{2}\delta^2\sum_{tt'}[2\hat{x}_t K_{tt'}\hat{w}_{t'} + \hat{x}_t C_{tt'}\hat{x}_{t'}]}\Big\rangle_{\{n\}} \\
&= \alpha\log\Big\langle \int \mathcal{D}\hat{w}\;\mathrm{e}^{-\frac{1}{2}\delta^2\sum_{tt'}\hat{w}_t L_{tt'}\hat{w}_{t'}} \\
&\quad\times \int \mathcal{D}w\;\mathrm{e}^{\mathrm{i}\delta\sum_{tt'} w_t(\mathbb{1}+EG)_{tt'}\hat{w}_{t'} - \frac{1}{2}\sum_{tt'} w_t(EDE)_{tt'}w_{t'}}\Big\rangle_{\{n\}} \\
&= \alpha\log\Big\langle\, \mathrm{Det}^{-\frac{1}{2}}[EDE]
\end{aligned}$$

$$\times \int \mathcal{D}\hat{w}\ \mathrm{e}^{-\frac{1}{2}\delta^2 \sum_{tt'} \hat{w}_t [L + (\mathbb{1}+G^\dagger E)(EDE)^{-1}(\mathbb{1}+EG)]_{tt'} \hat{w}_{t'}} \Big\rangle_{\{n\}}, \tag{5.35}$$

(apart from an irrelevant additive constant, consisting only of the logarithm of the various factors of 2π and δ which are generated by the final Gaussian integration over $\{w_t\}$). We finally work out the derivatives of this simplified expression (5.35) for Φ with respect to the matrix elements $\{L_{tt'}, C_{tt'}, G_{tt'}\}$, followed by setting $L \to 0$. Upon also using the causality property $G_{tt'} = 0$ for $t \le t'$, which guarantees that $\mathrm{Det}(\mathbb{1} + EG) = 1$, we find

$$\lim_{L\to 0} \frac{\partial \Phi}{\partial L_{tt'}} = -\frac{1}{2}\alpha \left\langle \left[(\mathbb{1}+EG)^{-1} EDE (\mathbb{1}+G^\dagger E)^{-1} \right]_{tt'} \right\rangle_{\{n\}}, \tag{5.36}$$

$$\lim_{L\to 0} \frac{\partial \Phi}{\partial C_{tt'}} = 0, \tag{5.37}$$

$$\lim_{L\to 0} \frac{\partial \Phi}{\partial G_{tt'}} = -\alpha \left\langle \left[(\mathbb{1}+EG)^{-1} E \right]_{t't} \right\rangle_{\{n\}}. \tag{5.38}$$

According to equation (5.31) this gives the following expressions for the entries of our conjugate order parameters

$$\hat{C}_{tt'} = 0, \quad \hat{K}_{tt'} = -\alpha R_{t't}, \quad \hat{L}_{tt'} = -\frac{1}{2} i \alpha \Sigma_{tt'}, \tag{5.39}$$

with the two matrices

$$\Sigma = \delta^{-2} \left\langle (\mathbb{1} + EG)^{-1} EDE (\mathbb{1} + G^\dagger E)^{-1} \right\rangle_{\{n\}}, \tag{5.40}$$

$$R = \delta^{-2} \left\langle (\mathbb{1} + EG)^{-1} E \right\rangle_{\{n\}}. \tag{5.41}$$

5.3.2 Evaluation of Poissonnian averages

It turns out that in the latter two matrices (5.40) and (5.41) the averages over the independently distributed Poissonnian random numbers $\{n_t\}$ can be performed exactly, by using causality of the response function G. We first deal with equation (5.41), which is somewhat simpler

$$R_{tt'} = \delta^{-2} \sum_{\ell \ge 0} (-1)^\ell \langle [(EG)^\ell E]_{tt'} \rangle_{\{n\}}. \tag{5.42}$$

The crucial observation is that, due to $G_{tt'} = 0$ for $t \le t'$, each of the terms in the above power series gives *factorized* Poissonnian averages, so that we only ever have to use the first Poissonnian moment $\langle n_t\rangle$. We recall that $E_{tt'}=\delta.\delta_{tt'}n_t/\langle n\rangle$ and work out the relevant matrix products

$$\begin{aligned}\langle[(EG)^\ell E]_{tt'}\rangle_{\{n\}} &= \sum_{s_1,\ldots,s_{\ell-1}}\langle(E_{tt}G_{ts_1})(E_{s_1s_1}G_{s_1s_2})\cdots(E_{s_{\ell-1}s_{\ell-1}}G_{s_{\ell-1}t'})E_{t't'}\rangle_{\{n\}}\\ &= \left(\frac{\delta}{\langle n\rangle}\right)^{\ell+1}\sum_{t>s_1>\cdots>s_{\ell-1}>t'} G_{ts_1}G_{s_1s_2}\cdots G_{s_{\ell-1}t'}\\ &\quad\times\langle n_t n_{s_1}\cdots n_{s_{\ell-1}}n_{t'}\rangle_{\{n\}}\\ &= \delta^{\ell+1}G^\ell_{tt'}. \end{aligned}\tag{5.43}$$

Insertion of equation (5.43) into the series expansion (5.42) for the kernel R leads to the surprisingly simple result

$$R=\delta^{-1}(\mathbb{1}+\delta G)^{-1}.\tag{5.44}$$

Next we turn to the matrix (5.40), where first we similarly write each of the terms $(\mathbb{1}+EG)^{-1}$ as a matrix power series, i.e.

$$\Sigma_{tt'}=\delta^{-2}\sum_{\ell\ell'\geq 0}(-1)^{\ell+\ell'}\langle[(EG)^\ell EDE(G^\dagger E)^{\ell'}]_{tt'}\rangle_{\{n\}}.\tag{5.45}$$

Here we no longer benefit from factorization of the Poissonnian averages in terms of first order moments. Instead of one ordered string, the causality of the response functions now generates two ordered strings of time indices. However, here we can have only averages of Poissonnian terms with at most *two* n-variables with the same time index. We thus have to take into account only the contributions coming from all possible pairings of time indices. Using $\langle n_t^2\rangle=\langle n_t\rangle^2[1+\tilde{\eta}/2\delta]$ we derive, again focusing on the individual terms in the matrix series (5.45):

$$\begin{aligned}&\langle[(EG)^\ell EDE(G^\dagger E)^{\ell'}]_{s_0s_0'}\rangle_{\{n\}}\\ &\quad=\sum_{s_\ell s_\ell'}D_{s_\ell s_\ell'}\langle[(EG)^\ell E]_{s_0s_\ell}[(EG)^{\ell'}E]_{s_0's_\ell'}\rangle_{\{n\}}\\ &\quad=\left(\frac{\delta}{\langle n\rangle}\right)^{\ell+\ell'+2}\sum_{s_0>\cdots>s_\ell}\ \sum_{s_0'>\cdots>s_\ell'}D_{s_\ell s_\ell'}\,G_{s_0s_1}\cdots G_{s_{\ell-1}s_\ell}\,G_{s_0's_1'}\cdots G_{s'_{\ell'-1}s_\ell'}\\ &\quad\times\langle n_{s_0}\cdots n_{s_\ell}n_{s_0'}\cdots n_{s_\ell'}\rangle_{\{n\}}.\end{aligned}$$

We note that in the above average $\langle\ldots\rangle_{\{n\}}$, each pairing of an index from the ordered string $\{s_0,\ldots,s_\ell\}$ with an index from the ordered string $\{s_0',\ldots,s_{\ell'}'\}$ produces a factor $\langle n^2\rangle=(1+\tilde{\eta}/2\delta)\langle n\rangle^2$, whereas each un-paired time index contributes simply a factor $\langle n\rangle$. Since the number of such pairings is $\sum_{i=0}^{\ell}\sum_{j=0}^{\ell'}\delta_{s_is_j'}$, we may write

$$
\begin{aligned}
&\langle[(EG)^{\ell}EDE(G^{\dagger}E)^{\ell'}]_{s_0s_0'}\rangle_{\{n\}} \\
&= \delta^{\ell+\ell'+2}\sum_{s_0>\cdots>s_\ell}\ \sum_{s_0'>\cdots>s_\ell'} D_{s_\ell s_\ell'}\left[1+\frac{\tilde{\eta}}{2\delta}\right]^{\sum_{i=0}^{\ell}\sum_{j=0}^{\ell'}\delta_{s_is_j'}} \\
&\quad\times G_{s_0s_1}\cdots G_{s_{\ell-1}s_\ell}\, G_{s_0's_1'}\cdots G_{s_{\ell'-1}'s_\ell'} \\
&= \delta^{\ell+\ell'+2}\sum_{s_0>\cdots>s_\ell}\ \sum_{s_0'>\cdots>s_\ell'} D_{s_\ell s_\ell'}\prod_{i=0}^{\ell}\prod_{j=0}^{\ell'}\left[1+\delta_{s_is_j'}\frac{\tilde{\eta}}{2\delta}\right] \\
&\quad\times G_{s_0s_1}\cdots G_{s_{\ell-1}s_\ell}\, G_{s_0's_1'}\cdots G_{s_{\ell'-1}'s_\ell'}.
\end{aligned}
\tag{5.46}
$$

5.3.3 Effective agent in the continuous time limit

With the results (5.39) we have arrived at a closed set of equations, involving only the order parameters $\{C,G\}$. The latter are to be solved from the two equations (5.29) and (5.30), which refer to a measure which is now simplified to

$$
\begin{aligned}
M[\{q,\hat{q},z\}] = p_0(q(0))\ &\mathrm{e}^{\mathrm{i}\delta\sum_t \hat{q}(t)\left[\frac{q(t+\delta)-q(t)}{\delta}-\delta^{-1}\theta(t)-\alpha\delta\sum_{t'}R_{tt'}\sigma[q(t'),z(t')]\right]} \\
&\times \mathrm{e}^{-\frac{1}{2}\alpha\delta^2\sum_{tt'}\hat{q}(t)\Sigma_{tt'}\hat{q}(t')}.
\end{aligned}
\tag{5.47}
$$

Since we may expand $R=\delta^{-1}\sum_{\ell\geq 0}(-\delta G)^{\ell}$, we know that $R_{tt'}=0$ for all $t<t'$. It now follows, similar to the reasoning which we followed for the batch MG,[24] that the present measure $M[\ldots]$ is normalized according to $\int\mathcal{D}q\mathcal{D}\hat{q}\ M[\{q,\hat{q},z\}]=1$. We may now replace the conjugate variable $\hat{q}(t')$ in equation (5.30) by differentiation with respect to the perturbation field $\theta(t')$, and thereby obtain order parameter equations that only involve the physical variables $\{q(t)\}$:

$$C_{tt'} = \langle\sigma[q(t),z(t)]\sigma[q(t'),z(t')]\rangle_\star, \tag{5.48}$$

$$G_{tt'} = \frac{\partial}{\partial\theta(t')}\langle\sigma[q(t),z(t)]\rangle_\star. \tag{5.49}$$

Upon integrating over the conjugate variables $\{\hat{q}(t)\}$ we find the corresponding effective measure $\langle g[\{q,z\}]\rangle_\star = \langle\int[\prod_t \mathrm{d}q(t)]\ M[\{q,z\}]g[\{q,z\}]\rangle_{\mathbf{z}}$, with

$$
\begin{aligned}
M[\{q,z\}] = p_0(q(0))\int\prod_t\left[\frac{\mathrm{d}\hat{q}(t)}{2\pi}\right]&\mathrm{e}^{-\frac{1}{2}\alpha\delta^2\sum_{tt'}\hat{q}(t)\Sigma_{tt'}\hat{q}(t')} \\
\times\mathrm{e}^{\mathrm{i}\delta\sum_t\hat{q}(t)\left[\frac{q(t+\delta)-q(t)}{\delta}-\delta^{-1}\theta(t)+\alpha\delta\sum_{t'}R_{tt'}\sigma[q(t'),z(t')]\right]}&
\end{aligned}
$$

[24] Again one integrates iteratively, starting with the variable $q(t_{\max})$, which gives $\delta[\hat{q}(t_{\max})]$, and so on until one reaches $t=0$, where one uses the normalization of $p_0(q(0))$.

$$
\begin{aligned}
&= p_0(q(0)) \int \prod_t \left[\frac{\mathrm{d}\eta(t) d\hat{q}(t)}{2\pi}\right] \mathrm{e}^{\mathrm{i}\sqrt{\alpha}\delta \sum_t \hat{q}(t)\eta(t) - \frac{1}{2}\alpha\delta^2 \sum_{tt'} \hat{q}(t)\Sigma_{tt'}\hat{q}(t')} \\
&\quad \times \prod_t \delta\left[\eta(t) - \frac{(q(t+\delta)-q(t))/\delta + \theta(t)/\delta - \alpha\delta \sum_{t'} R_{tt'}\sigma[q(t'),z(t')]}{\sqrt{\alpha}}\right] \\
&= p_0(q(0)) \int \prod_t \left[\frac{\mathrm{d}\eta(t)}{\delta\sqrt{2\pi}}\right] \frac{\mathrm{e}^{-\frac{1}{2}\sum_{tt'} \eta(t)\Sigma^{-1}_{tt'}\eta(t')}}{\sqrt{\mathrm{Det}\Sigma}} \\
&\quad \times \prod_{t\geq 0} \delta\left[\frac{q(t+\delta)-q(t)}{\delta} - \frac{\theta(t)}{\delta} + \alpha\delta \sum_{t'} R_{tt'}\sigma[q(t'),z(t')] - \sqrt{\alpha}\eta(t)\right].
\end{aligned}
\tag{5.50}
$$

Here $R_{tt'} = \delta^{-1}[\mathbf{1} + \delta G]^{-1}_{tt'}$, and $\Sigma_{tt'}$ is given by equations (5.40) and (5.46). As in the case of the batch MG we finally end up with a mathematical representation of our measure which describes an effective single-agent process. Here this process is found to be defined as follows, with $\tilde{\theta}(t) = \theta(t)/\delta$:

$$
\frac{q(t+\delta)-q(t)}{\delta} = \tilde{\theta}(t) - \alpha\delta \sum_{t'\leq t} R_{tt'}\sigma[q(t'),z(t')] + \sqrt{\alpha}\eta(t) \tag{5.51}
$$

in which the disorder-generated zero-averages Gaussian noise variables $\eta(t)$ are characterized by the covariance matrix $\langle \eta(t)\eta(t')\rangle_\star = \Sigma_{tt'}$. The averages $\langle \ldots \rangle_\star$ in equations (5.48) and (5.49), from which we have to solve our dynamic order parameters $\{C, G\}$, now refer to averaging over the process (5.51) and over the Gaussian decision noise variables $\{z(t)\}$.

Finally we can take the limit $\delta \to 0$ and restore continuous time. The bookkeeping of δ-terms is found to come out right in the above equations, with partial derivatives with respect to the perturbation fields converting into functional derivatives, with time summations converting into integrals, and with matrices converting into integral operators (with the usual convention $\mathbf{1}_{tt'} \to \delta.\delta[t-t']$). Upon making the ansatz that our order parameters are smooth functions of time, we also lose the remaining dependence on the microscopic variables $\{z(t)\}$ in the retarded self-interaction term. The latter is automatically converted into an average over their distribution, and (with the usual definition $\sigma[q] = \int \mathrm{d}z\, P(z)\sigma[q,z]$) we end up with the following appealing continuous-time effective single-agent equation

$$
\frac{\mathrm{d}}{\mathrm{d}t} q(t) = \tilde{\theta}(t) - \alpha \int_0^t \mathrm{d}t'\, R(t,t')\sigma[q(t')] + \sqrt{\alpha}\eta(t). \tag{5.52}
$$

The correlation and response functions (5.23) and (5.24), the final dynamic order parameters of the problem, have become kernels with two real-valued time-arguments, and are to be solved self-consistently from

$$C(t,t') = \langle \sigma[q(t)]\sigma[q(t')]\rangle_\star, \quad G(t,t') = \frac{\delta}{\delta\tilde{\theta}(t')}\langle \sigma[q(t)]\rangle_\star \tag{5.53}$$

(for $t \neq t'$). The brackets $\langle \ldots \rangle_\star$ now refer to the stochastic process (5.52), which no longer involves the $\{z(t)\}$. We note that the correlation- and response functions are generally discontinuous at $t = t'$, viz. $C(t,t) = 1$ and $G(t,t) = 0$ (in our derivation we have effectively used the so-called Itô convention).

Our remaining task is to take the continuous time limit in our expressions for the effective noise covariance Σ and the retarded self-interaction kernel R. The latter, defined by equation (5.44), simply becomes

$$R(t,t') = \lim_{\delta\to 0} \delta^{-1}[\mathbb{1} + \delta G]^{-1}_{tt'} = \delta(t-t') + \sum_{\ell>0}(-1)^\ell G^\ell(t,t') \tag{5.54}$$

with the conventional definition for multiplication of the continuous time kernels, viz. $(AB)(t,t') = \int \mathrm{d}s\, A(t,s)B(s,t')$. The continuous time limit of the less trivial noise covariance (5.40) and (5.46) becomes

$$\begin{aligned}\Sigma(s_0,s_0') &= \sum_{\ell\ell'\geq 0}(-1)^{\ell+\ell'}\lim_{\delta\to 0}\delta^{\ell+\ell'}\sum_{s_0>\cdots>s_\ell\geq 0}\;\sum_{s_0'>\cdots>s_{\ell'}'\geq 0}\prod_{i=0}^{\ell}\prod_{j=0}^{\ell'}\left[1+\delta_{s_i s_j'}\frac{\tilde{\eta}}{2\delta}\right]\\ &\quad\times[1+C(s_\ell,s_{\ell'}')]\;G(s_0,s_1)\cdots G(s_{\ell-1},s_\ell)G(s_0',s_1')\cdots G(s_{\ell'-1}',s_{\ell'}')\\ &= \sum_{\ell\ell'\geq 0}(-1)^{\ell+\ell'}\int_0^\infty \mathrm{d}s_1,\ldots \mathrm{d}s_\ell, \mathrm{d}s_1',\ldots,\mathrm{d}s_{\ell'}'\prod_{i=0}^{\ell}\prod_{j=0}^{\ell'}\left[1+\frac{1}{2}\tilde{\eta}\delta(s_i-s_j')\right]\\ &\quad\times[1+C(s_\ell,s_{\ell'}')]\;G(s_0,s_1),\ldots,G(s_{\ell-1},s_\ell),G(s_0',s_1'),\ldots,G(s_{\ell'-1}',s_{\ell'}').\end{aligned} \tag{5.55}$$

We may conclude from the above results that the decision noise fluctuations have vanished from our theory. The decision noise will still influence the process, but only via the noise-averaged function $\sigma[q]$. We would therefore in retrospect have been allowed to average our microscopic laws over the $\{z(t)\}$ without punishment. However, this is not the case for the fluctuations induced by the random selections of the 'fake memory' variables $\{\mu(\ell)\}$. Averaging the microscopic laws over these latter variables, which was the basis of the pseudo-equilibrium calculation in Chapter 3, is equivalent to taking the limit $\tilde{\eta} \to 0$. This can lead at most to an approximate dynamical theory, for $\tilde{\eta}$ is still very much present in equation (5.55). We will come back to the impact of such approximations later.

5.4 Stationary state in the ergodic regime

In this section we study the long-time limit of the effective single-agent process (5.52), upon assuming that the stationary state has the familiar simplifying properties of time-translation invariance and a finite susceptibility, i.e. $C(t,t')=C(t-t')$, $G_{tt'}=G(t-t')$, and $\chi=\int_0^\infty \mathrm{d}t\, G(t)<\infty$. Since this implies also that C and G commute, we can be sure that also the kernels R and Σ depend on time differences only: $R(t,t')=R(t-t')$ and $\Sigma(t,t')=\Sigma(t-t')$. A simple generalization of the arguments applied in the case of the batch MG, whereby all summations are replaced by integrals via $\sum_t \to \delta^{-1}\int \mathrm{d}t$, reveals that

$$\int \mathrm{d}t'\ R(t-t') = \int \mathrm{d}t' \left\{\delta(t-t') + \sum_{\ell>0}(-1)^\ell G^\ell(t-t')\right\}$$
$$= 1 + \sum_{\ell>0}(-1)^\ell \chi^\ell = \frac{1}{1+\chi}. \tag{5.56}$$

In this section, in line with our previous notation conventions, we will write time averages as $\overline{x}=\lim_{\tau\to\infty}\tau^{-1}\int_0^\tau dt\, x(t)$.

5.4.1 Closed laws for persistent order parameters

We can adapt the procedures that had been successful in the case of the batch MG to our continuous time equation (5.52), simply by replacing time summations by time integrals. We expect qualitative differences to emerge at most in the evaluation of those terms that involve the kernel Σ, in view of the dependence of Σ (5.55) on the learning rate $\tilde{\eta}$, which was absent in the batch dynamics. Upon defining the transformed variables $\tilde{q}(t)=q(t)/t$ we find the integrated form of equation (5.52) in the absence of external perturbations converting into

$$\tilde{q}(t) = \frac{q(0)}{t} + \frac{1}{t}\int_0^t \mathrm{d}t' \left\{\sqrt{\alpha}\eta(t') - \alpha\int_0^{t'} \mathrm{d}s\ R(t',s)\sigma[s\tilde{q}(s)]\right\}. \tag{5.57}$$

Provided the limit $\tilde{q}=\lim_{t\to\infty}\tilde{q}(t)$ exists, we may send $t\to\infty$ in equation (5.57) and recover exactly the corresponding statement (4.120) for the batch MG

$$\tilde{q} = \sqrt{\alpha}\,\overline{\eta} - \frac{\alpha\overline{\sigma}}{1+\chi}, \tag{5.58}$$

with the zero-average Gaussian variable $\overline{\eta}=\lim_{\tau\to\infty}\tau^{-1}\int_0^\tau \mathrm{d}t\ \eta(t)$ and with $\overline{\sigma}=\lim_{\tau\to\infty}\tau^{-1}\int_0^\tau \mathrm{d}t\ \sigma[t\tilde{q}(t)]=\mathrm{sgn}[\tilde{q}].\sigma[\infty]$. We conclude that we may copy the classification into fickle and frozen agents as found for the batch model, i.e.

$$|\overline{\eta}| \le \frac{\sigma[\infty]\sqrt{\alpha}}{1+\chi}: \quad \text{'fickle' solution,} \qquad \overline{\sigma} = \frac{(1+\chi)\overline{\eta}}{\sqrt{\alpha}}, \tag{5.59}$$

$$|\overline{\eta}| > \frac{\sigma[\infty]\sqrt{\alpha}}{1+\chi}: \quad \text{'frozen' solution,} \quad \overline{\sigma} = \sigma[\infty]\mathrm{sgn}[\overline{\eta}]. \tag{5.60}$$

Also the self-consistent calculation of the fraction of frozen agents ϕ, the susceptibility χ, and the persistent correlations c can be largely copied from the batch version, with the only proviso that we may no longer substitute the batch value for the variance $\langle\overline{\eta}^2\rangle_\star$ but limit ourselves for now to expressing $\{c, \phi, \chi\}$ in terms of the as yet unknown present value of $\langle\overline{\eta}^2\rangle_\star$. The basis of our calculations is again to distinguish between frozen and fickle solutions, following equations (5.59) and (5.60). Upon defining

$$v = \frac{\sigma[\infty]\sqrt{\alpha}}{\sqrt{2}(1+\chi)\langle\overline{\eta}^2\rangle^{1/2}} \tag{5.61}$$

we find for $\{c, \phi, \chi\}$ exactly the corresponding batch MG equations

$$\begin{aligned}
c &= \int \mathrm{d}\overline{\eta}\; P(\overline{\eta})\overline{\sigma}^2 \\
&= \frac{\sigma^2[\infty]}{2v^2}\int Dz\; z^2\theta[v\sqrt{2}-|z|] + \sigma^2[\infty]\int Dz\;\theta[|z|-v\sqrt{2}] \\
&= \sigma^2[\infty]\Big\{1+\frac{1-2v^2}{2v^2}\mathrm{Erf}[v] - \frac{1}{v\sqrt{\pi}}\mathrm{e}^{-v^2}\Big\},
\end{aligned} \tag{5.62}$$

$$\begin{aligned}
\phi &= \int \mathrm{d}\overline{\eta}\; P(\overline{\eta})\;\theta\Big[|\overline{\eta}| - \frac{\sigma[\infty]\sqrt{\alpha}}{1+\chi}\Big] \\
&= \int Dz\;\theta[|z|-v\sqrt{2}] = 1-\mathrm{Erf}[v]
\end{aligned} \tag{5.63}$$

$$\begin{aligned}
\chi &= \frac{1}{\sqrt{\alpha}}\langle\frac{\partial}{\partial\overline{\eta}}\overline{\sigma}\rangle_\star = \frac{1}{\sqrt{\alpha}}\int \mathrm{d}\overline{\eta}\; P(\overline{\eta})\frac{\partial\overline{\sigma}}{\partial\overline{\eta}} \\
&= \frac{1+\chi}{\alpha}\int Dz\;\frac{\mathrm{d}}{\mathrm{d}z}\Big\{z\;\theta[v\sqrt{2}-|z|] + \mathrm{sgn}[z]\;\theta[|z|-v\sqrt{2}]\;v\sqrt{2}\Big\} \\
&= \frac{1+\chi}{\alpha}\mathrm{Erf}[v].
\end{aligned} \tag{5.64}$$

The last line is easily rewritten in the familiar form $\chi = (1-\phi)/(\alpha-1+\phi)$. The only possible difference between the on-line and the batch calculation (at least, with regard to persistent order parameters in TTI stationary states) therefore lies in the value taken by the variance of the noise variable $\overline{\eta}$. Here we find the more complicated formula

$$\begin{aligned}
\langle\overline{\eta}^2\rangle_\star &= \lim_{\tau\to\infty}\frac{1}{\tau^2}\int_0^\tau \mathrm{d}t\mathrm{d}t'\;\Sigma(t-t') \\
&= \sum_{\ell\ell'\geq 0}(-1)^{\ell+\ell'}\lim_{\tau\to\infty}\frac{1}{\tau^2}\int_0^\tau \mathrm{d}s_0\mathrm{d}s_0'\int_0^\infty \mathrm{d}s_1,\ldots,\mathrm{d}s_\ell\mathrm{d}s_1',\ldots,\mathrm{d}s_\ell' \\
&\quad\times[1+C(s_\ell-s_\ell')]\prod_{i=0}^{\ell}\prod_{j=0}^{\ell'}\Big[1+\frac{1}{2}\tilde{\eta}\delta(s_i-s_j')\Big]
\end{aligned}$$

$$\times\, G(s_0 - s_1), \ldots, G(s_{\ell-1} - s_\ell)\; G(s'_0 - s'_1), \ldots G(s'_{\ell'-1} - s'_\ell). \quad (5.65)$$

5.4.2 Equivalence of batch and on-line equations

We next set out to show that the above expression (5.65) is *independent* of $\tilde{\eta}$ (given the assumptions of the present section). First we transform our time variables: we choose $t_\ell = s_\ell$, and $t_k = s_k - s_{k+1}$ for all $k < \ell$, with similar definitions for the times t'_k. Equivalently: $s_k = \sum_{r=k}^{\ell} t_r$ for all $k = 0, \ldots, \ell$ and $s'_k = \sum_{r=k}^{\ell'} t'_r$ for all $k = 0, \ldots, \ell'$. This allows us to write expression (5.65) as

$$\begin{aligned}\langle \overline{\eta}^2 \rangle_\star = &\sum_{\ell\ell' \geq 0} (-1)^{\ell+\ell'} \int_0^\infty \mathrm{d}t_0, \ldots, \mathrm{d}t_{\ell-1}\, G(t_0), \ldots, G(t_{\ell-1}) \\ &\times \int_0^\infty \mathrm{d}t'_0, \ldots, \mathrm{d}t'_{\ell'-1}\, G(t'_0), \ldots, G(t'_{\ell'-1}) \\ &\times \lim_{\tau\to\infty} \int_0^{\tau - \sum_{r=0}^{\ell-1} t_r} \frac{\mathrm{d}t_\ell}{\tau} \int_0^{\tau - \sum_{r=0}^{\ell'-1} t'_r} \frac{\mathrm{d}t'_{\ell'}}{\tau}\, [1 + C(t_\ell - t'_{\ell'})] \\ &\times \prod_{i=0}^{\ell} \prod_{j=0}^{\ell'} \left[1 + \frac{1}{2}\tilde{\eta}\delta\Big[\sum_{r=i}^{\ell} t_r - \sum_{r=j}^{\ell'} t'_r\Big]\right].\end{aligned} \quad (5.66)$$

The last line can be expanded to give unity plus a sum of products of δ-distributions. However, we see, since $\lim_{t\to\infty} \int_0^t \mathrm{d}t'\, G(t') = \chi < \infty$, that every term which contains one or more of these δ-distributions effectively removes at the very least the summation over t_ℓ (since t_ℓ will then be given one allowed value only), so that the corresponding normalization factor τ^{-1} is no longer compensated and the limit $\tau \to \infty$ produces a zero result. Hence

$$\begin{aligned}\langle \overline{\eta}^2 \rangle_\star = &\sum_{\ell\ell' \geq 0} (-1)^{\ell+\ell'} \int_0^\infty \mathrm{d}t_0, \ldots, \mathrm{d}t_{\ell-1}\, G(t_0), \ldots, G(t_{\ell-1}) \\ &\times \int_0^\infty \mathrm{d}t'_0, \ldots, \mathrm{d}t'_{\ell'-1}\, G(t'_0), \ldots, G(t'_{\ell'-1}) \\ &\times \lim_{\tau\to\infty} \int_0^{\tau - \sum_{r=0}^{\ell-1} t_r} \frac{\mathrm{d}t_\ell}{\tau} \int_0^{\tau - \sum_{r=0}^{\ell'-1} t'_r} \frac{\mathrm{d}t'_{\ell'}}{\tau}\, [1 + C(t_\ell - t'_{\ell'})] \\ = &\,(1+c) \sum_{\ell\ell' \geq 0} (-1)^{\ell+\ell'} \chi^{\ell+\ell'} = \frac{1+c}{(1+\chi)^2}.\end{aligned} \quad (5.67)$$

This proves our statement that the learning rate $\tilde{\eta}$ indeed drops out. From this, in turn, it follows that the equations describing TTI states in the on-line MG have become fully *identical* to both those of the batch MG in Chapter 4 and to those obtained via the replica calculation in Chapter 3. This includes the locations of the phase transition lines (defined by the divergence of χ), and the full behaviour of the persistent observables $\{c, \phi, \chi\}$ in the ergodic regime $\alpha > \alpha_c$, for both additive and multiplicative decision noise. We have thereby confirmed that these earlier equations are exact, and that in the ergodic regime there are no differences between batch and on-line dynamics in the behaviour of $\{c, \phi, \chi\}$, which explains these somewhat puzzling earlier findings.

There is obviously no further need to carry out numerical simulations to test the predictions of the present stationary state theory of the on-line MG against simulations, since this has already been done in Chapter 3 (albeit that at the time we could not yet be sure of the validity of these equations).

However, our conclusion that in the ergodic regime of the on-line MG the persistent observables $\{c, \phi, \chi\}$ take the same values as in the batch MG and the fluctuation-free MG, does not imply that the same is true for the volalitity σ.

5.4.3 Overall bid statistics and volatility

Let us first derive exact expressions for the volatility matrix in the on-line MG. As in the batch MG, viz. (4.57), we can extract the global bid statistics from an appropriate generating functional. If we add the new generating fields to the temporally regularized microscopic process, we obtain, with $t \in \{0, \delta, 2\delta, \ldots\}$:

$$Z[\boldsymbol{\phi}] = \langle \mathrm{e}^{\mathrm{i}\sqrt{2}\sum_{t\geq 0}\sum_{\mu}\phi_\mu(t)A^\mu[\mathbf{q}(t),\mathbf{z}(t)]} \rangle. \tag{5.68}$$

From equation (5.68) one obtains as before

$$\overline{\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]\rangle} = -\frac{i}{\sqrt{2}} \lim_{\boldsymbol{\phi}\to\mathbf{0}} \frac{\partial \overline{Z[\boldsymbol{\phi}]}}{\partial \phi_\mu(t)}, \tag{5.69}$$

$$\overline{\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]A^\nu[\mathbf{q}(t'),\mathbf{z}(t')]\rangle} = -\frac{1}{2} \lim_{\boldsymbol{\phi}\to\mathbf{0}} \frac{\partial^2 \overline{Z[\boldsymbol{\phi}]}}{\partial \phi_\mu(t)\partial\phi_\nu(t')}. \tag{5.70}$$

We thus find the new generating functional $Z[\boldsymbol{\phi}]$ by making in expression (5.15) for the previous generating functional $Z[\boldsymbol{\psi}]$ the following substitution

$$\mathrm{e}^{\mathrm{i}\delta\sum_{it}\psi_i(t)s_i(t)} \;\longrightarrow\; \mathrm{e}^{\mathrm{i}\sum_{t\mu}\phi_\mu(t)[\sqrt{2}\Omega_\mu + x_t^\mu]}.$$

At this point the present calculation starts to differ from that of the batch version, in reflection of the fake memory fluctuations in the on-line MG. Here we find, with $\mathcal{D}\mathbf{q} = \prod_{it}[\mathrm{d}q_i(t)/\sqrt{\delta/2\pi}]$ (and similarly for the other variables)

$$Z[\boldsymbol{\phi}] = \int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\; \mathrm{e}^{\delta\sum_{t\mu}\left[\mathrm{i}(\hat{w}_t^\mu w_t^\mu + \hat{x}_t^\mu x_t^\mu) + \mathrm{i}\delta^{-1}\phi_\mu(t)(\sqrt{2}\Omega_\mu + x_t^\mu)\right]}$$

$$\times\ \mathrm{e}^{\frac{2}{\tilde{\eta}}\delta\sum_{t\mu}[\mathrm{e}^{\frac{1}{2}i\tilde{\eta}w_t^\mu(\sqrt{2}\Omega_\mu+x_t^\mu)}-1+\epsilon_N]}$$
$$\times \int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}}\, p_0(\mathbf{q}(0))\, \mathrm{e}^{\mathrm{i}\delta\sum_{ti}\hat{q}_i(t)[q_i(t+\delta)-q_i(t)-\theta_i(t)]}$$
$$\times\ \langle \mathrm{e}^{-\frac{\mathrm{i}\sqrt{2}\delta}{\sqrt{N}}\sum_{\mu i}\xi_i^\mu\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]}\rangle_{\mathbf{z}}. \tag{5.71}$$

For initial conditions of the form $p_0(\mathbf{q}) = \prod_i p_0(q_i)$, and upon following closely the steps which we took when deriving our previous Eqs. (5.17) and (5.18) in the evaluation of $\overline{Z[\boldsymbol{\psi}]}$, we obtain the disorder-averaged expression

$$\overline{Z[\boldsymbol{\phi}]} = \int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\, \mathrm{e}^{\mathrm{i}\delta\sum_{t\mu}[\hat{w}_t^\mu w_t^\mu+\hat{x}_t^\mu x_t^\mu+\delta^{-1}\phi_\mu(t)x_t^\mu]} \int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}}\, \prod_i p_0(q_i(0))$$
$$\times\ \mathrm{e}^{\mathrm{i}\delta\sum_{ti}\hat{q}_i(t)[q_i(t+\delta)-q_i(t)-\theta_i(t)]}$$
$$\times \int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\, \mathrm{e}^{\mathrm{i}N\delta^2\sum_{tt'}[\hat{C}_{tt'}C_{tt'}+\hat{K}_{tt'}K_{tt'}+\hat{L}_{tt'}L_{tt'}]}$$
$$\times\ \mathrm{e}^{-\mathrm{i}\delta^2\sum_i\sum_{tt'}[\hat{C}_{tt'}s_i(t)s_i(t')+\hat{K}_{tt'}s_i(t)\hat{q}_i(t')+\hat{L}_{tt'}\hat{q}_i(t)\hat{q}_i(t')]}$$
$$\times \prod_\mu \Bigg\{ \int \frac{\mathrm{d}\Omega}{\sqrt{\pi}}\, \mathrm{e}^{-\Omega^2+\mathrm{i}\sqrt{2}\Omega\sum_t\phi_\mu(t)+\frac{2\delta}{\tilde{\eta}}\sum_t[\mathrm{e}^{\frac{1}{2}i\tilde{\eta}w_t^\mu(\sqrt{2}\Omega+x_t^\mu)}-1+\epsilon_N]+\mathcal{O}(N^{-1})}$$
$$\times\ \mathrm{e}^{-\frac{1}{2}\delta^2\sum_{tt'}\left[\hat{w}_t^\mu L_{tt'}\hat{w}_{t'}^\mu+2\hat{x}_t^\mu K_{tt'}\hat{w}_{t'}^\mu+\hat{x}_t^\mu C_{tt'}\hat{x}_{t'}^\mu\right]}\Bigg\}. \tag{5.72}$$

The result is again an integral which for $N\to\infty$ can be evaluated by steepest descent

$$\overline{Z[\boldsymbol{\phi}]} = \int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\; e^{N[\Psi+\Phi+\Omega+\tilde{\epsilon}_N]}, \tag{5.73}$$

but now with

$$\Psi = \mathrm{i}\delta^2\sum_{tt'}[\hat{C}_{tt'}C_{tt'} + \hat{K}_{tt'}K_{tt'} + \hat{L}_{tt'}L_{tt'}], \tag{5.74}$$

$$\Phi = \frac{1}{N}\sum_\mu \log\Bigg[\int \mathcal{D}w\mathcal{D}\hat{w}\mathcal{D}x\mathcal{D}\hat{x}\; e^{-\frac{1}{2}\delta^2\sum_{tt'}[\hat{w}_tL_{tt'}\hat{w}_{t'}+2\hat{x}_tK_{tt'}\hat{w}_{t'}+\hat{x}_tC_{tt'}\hat{x}_{t'}]}$$
$$\times\ \mathrm{e}^{\mathrm{i}\delta\sum_t[\hat{w}_tw_t+\hat{x}_tx_t+\delta^{-1}\phi_\mu(t)x_t^\mu]}\int Du\prod_t \mathrm{e}^{\mathrm{i}u\phi_\mu(t)+\frac{2\delta}{\tilde{\eta}}[\mathrm{e}^{\frac{1}{2}i\tilde{\eta}w_t(u+x_t)}-1]}\Bigg] \tag{5.75}$$

$$\Omega = \frac{1}{N}\sum_i \log\langle \int \mathcal{D}q\mathcal{D}\hat{q}\; p_0(q(0))\; \mathrm{e}^{\mathrm{i}\delta\sum_t\hat{q}(t)[\frac{q(t+\delta)-q(t)}{\delta}-\delta^{-1}\theta_i(t)]}$$
$$\times\ \mathrm{e}^{-\mathrm{i}\delta^2\sum_{tt'}[\hat{C}_{tt'}\sigma[q(t),z(t)]\sigma[q(t'),z(t')]+\hat{K}_{tt'}\sigma[q(t),z(t)]\hat{q}(t')+\hat{L}_{tt'}\hat{q}(t)\hat{q}(t')]}\rangle_{\mathbf{z}}. \tag{5.76}$$

As in the case of the batch MG, for $\boldsymbol{\psi}\to\mathbf{0}$ the differences between $\overline{Z[\boldsymbol{\psi}]}$ and the present $\overline{Z[\boldsymbol{\phi}]}$ are concentrated in the function Φ. This allows us again to use the property $L = 0$ at the saddle-point, and the previous transformations $K = iG$ (with the response function G) and $D_{tt'} = 1 + C_{tt'}$ (with the correlation function C) for simplifying Φ at the saddle-point. For the on-line MG this is obviously more involved, and with the specific notation conventions of Section 5.3.1 (especially the function $\phi[\{x, w\}, u]$ and its representation in terms of Poissonnian averages with $\langle n \rangle = 2\delta/\tilde{\eta}$, and the matrix E) we find

$$\begin{aligned}
\Phi &= \frac{1}{N}\sum_{\mu} \log \left[\int \mathcal{D}w\mathcal{D}\hat{w}\mathcal{D}x\mathcal{D}\hat{x}\ \mathrm{e}^{-\frac{1}{2}\delta^2 \sum_{tt'}[2i\hat{x}_t G_{tt'}\hat{w}_{t'} + \hat{x}_t C_{tt'}\hat{x}_{t'}]} \right. \\
&\quad \left. \times\ \mathrm{e}^{\mathrm{i}\delta \sum_t [\hat{w}_t w_t + \hat{x}_t x_t + \delta^{-1}\phi_\mu(t) x_t^\mu]} \int Du\ \phi[\{x, w\}, u] \mathrm{e}^{\mathrm{i}u \sum_t \phi_\mu(t)} \right] \\
&= \frac{1}{N}\sum_{\mu} \log \left\langle \int \mathcal{D}w\mathcal{D}\hat{w}\mathcal{D}x\mathcal{D}\hat{x}\ \mathrm{e}^{\mathrm{i}\delta \sum_t \hat{w}_t [w_t - \delta \sum_{t'} G_{t't}\hat{x}_{t'}] - \frac{1}{2}\delta^2 \sum_{tt'} \hat{x}_t C_{tt'}\hat{x}_{t'}} \right. \\
&\quad \left. \times\ \mathrm{e}^{\mathrm{i}\delta \sum_t x_t [\hat{x}_t + \delta^{-1}\phi_\mu(t) + \frac{n_t w_t}{\langle n \rangle}]} \int Du\ \mathrm{e}^{\mathrm{i}u\delta \sum_t [\delta^{-1}\phi_\mu(t) + \frac{n_t w_t}{\langle n \rangle}]} \right\rangle_{\{n\}} \\
&= \frac{1}{N}\sum_{\mu} \log \left\langle \int \mathcal{D}w\mathcal{D}\hat{w}\ \mathrm{e}^{-\frac{1}{2}\sum_{tt'}[\phi_\mu(t) + \sum_s E_{ts} w_s] D_{tt'} [\phi_\mu(t') + \sum_s E_{t's} w_s]} \right. \\
&\quad \left. \times\ \mathrm{e}^{\mathrm{i}\delta \sum_t \hat{w}_t w_t + i\delta \sum_{tt'}[\phi_\mu(t) + \sum_s E_{ts} w_s] G_{tt'} \hat{w}_{t'}} \right\rangle_{\{n\}} \\
&= \frac{1}{N}\sum_{\mu} \log \left\langle \prod_t \left[\frac{\mathrm{d}y_t}{\sqrt{2\pi}} \right] \mathrm{e}^{-\frac{1}{2}\sum_{tt'} y_t [(\mathbf{1} + G^\dagger E)(EDE)^{-1}(\mathbf{1} + EG)]_{tt'} y_{t'}} \right. \\
&\quad \left. \times\ \mathrm{e}^{-\mathrm{i}\sum_t y_t (E^{-1})_{tt'} \phi_\mu(t')} \right\rangle_{\{n\}} + \alpha \log \langle \mathrm{Det}^{-\frac{1}{2}}[EDE] \rangle_{\{n\}} \\
&= \frac{1}{N}\sum_{\mu} \log \left\langle \mathrm{e}^{-\frac{1}{2}\sum_{tt'} \phi_\mu(t) [E^{-1}(\mathbf{1} + EG)^{-1} EDE (\mathbf{1} + G^\dagger E)^{-1} E^{-1}]_{tt'} \phi_\mu(t')} \right\rangle_{\{n\}} \\
&\quad - \alpha \log \langle \mathrm{Det}[\mathbf{1} + G^\dagger E] \rangle_{\{n\}}.
\end{aligned} \tag{5.77}$$

The last line can be simplified, using the identity $E^{-1}(\mathbf{1} + EG)^{-1} = (\mathbf{1} + GE)^{-1}E^{-1}$ and using the fact that we will only need this expression for operators G which obey causality, to

$$\Phi = \frac{1}{N}\sum_{\mu} \log \left\langle \mathrm{e}^{-\frac{1}{2}\sum_{tt'} \phi_\mu(t) [(\mathbf{1} + GE)^{-1} D (\mathbf{1} + EG^\dagger)^{-1}]_{tt'} \phi_\mu(t')} \right\rangle_{\{n\}}. \tag{5.78}$$

We may now calculate the derivatives required to work out equations (5.69) and (5.70).

$$\lim_{\boldsymbol{\phi} \to \mathbf{0}} \frac{\partial N\Phi}{\partial \phi_\mu(t)} = 0. \tag{5.79}$$

$$\lim_{\boldsymbol{\phi}\to\mathbf{0}} \frac{\partial^2 N\Phi}{\partial\phi_\mu(t)\partial\phi_\nu(t')} = -\delta_{\mu\nu}\Big\langle [(\mathbb{1}+GE)^{-1}D(\mathbb{1}+EG^\dagger)^{-1}]_{tt'}\Big\rangle_{\{n\}}, \quad (5.80)$$

and find, with our usual shorthand $\{\mathcal{D}\} = \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}K\mathcal{D}\hat{K}\mathcal{D}L\mathcal{D}\hat{L}$ and with the normalization identity $Z[\mathbf{0}] = 1$, that

$$\begin{aligned}\lim_{N\to\infty} \overline{\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]\rangle} &= -\frac{i}{\sqrt{2}} \lim_{N\to\infty}\lim_{\boldsymbol{\phi}\to\mathbf{0}} \frac{\int\{\mathcal{D}\}(\partial/\partial\phi_\mu(t))\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\ &= 0 \end{aligned} \quad (5.81)$$

$$\begin{aligned}&\lim_{N\to\infty} \overline{\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]A^\nu[\mathbf{q}(t'),\mathbf{z}(t')]\rangle} \\ &\quad = -\frac{1}{2}\lim_{N\to\infty}\lim_{\boldsymbol{\phi}\to\mathbf{0}} \frac{\int\{\mathcal{D}\}(\partial^2/\partial\phi_\mu(t)\partial\phi_\nu(t'))\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\,\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\ &\quad = \frac{1}{2}\delta_{\mu\nu}\Big\langle[(\mathbb{1}+GE)^{-1}D(\mathbb{1}+EG^\dagger)^{-1}]_{tt'}\Big\rangle_{\{n\}}. \end{aligned} \quad (5.82)$$

Upon making our usual assumption that $p^{-1}\sum_{\mu=1}^{p}\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]\rangle$ will be self-averaging for $N\to\infty$ (i.e. no longer dependent on the microscopic realization of the disorder), we see that in the on-line MG the asymptotic disorder-averaged volatility matrix is given by

$$\begin{aligned}\overline{\Xi}_{tt'} &= \lim_{N\to\infty}\frac{1}{p}\sum_{\mu=1}^{p}\overline{\langle A^\mu[\mathbf{q}(t),\mathbf{z}(t)]A^\mu[\mathbf{q}(t'),\mathbf{z}(t')]} \\ &= \frac{1}{2}\langle[(\mathbb{1}+GE)^{-1}D(\mathbb{1}+EG^\dagger)^{-1}]_{tt'}\rangle_{\{n\}}. \end{aligned} \quad (5.83)$$

Comparison with equations (5.40) and (5.46) shows that, although similar, this expression is not identical to the covariance matrix of the effective noise $\{\eta(t)\}$ in the single-agent equation, in contrast to the situation in the batch MG.

Let us finally work out the Poissonnian averages in equation (5.83) and take the continuous time limit $\delta\to 0$, similar to how this was done for the noise covariances $\Sigma(s_0, s_0')$. We first write

$$\begin{aligned}2\overline{\Xi}_{s_0 s_0'} &= \sum_{\ell\ell'\geq 0}(-1)^{\ell+\ell'}\langle[(GE)^\ell D(EG^\dagger)^{\ell'}]_{s_0 s_0'}\rangle_{\{n\}} \\ &= \sum_{\ell\ell'\geq 0}\left(\frac{-\delta}{\langle n\rangle}\right)^{\ell+\ell'}\sum_{s_0>\cdots>s_\ell}\sum_{s_0'>\cdots>s_\ell'} D_{s_\ell s_\ell'}G_{s_0 s_1},\ldots,G_{s_{\ell-1}s_\ell}G_{s_0' s_1'},\ldots,G_{s_{\ell'-1}' s_\ell'} \\ &\qquad \times\langle n_{s_1},\ldots,n_{s_\ell}n_{s_1'},\ldots,n_{s_\ell'}\rangle_{\{n\}}. \end{aligned}$$

In the average $\langle \ldots \rangle_{\{n\}}$, each pairing of an index from the set $\{s_0, \ldots, s_\ell\}$ with one from the set $\{s'_0, \ldots, s'_{\ell'}\}$ gives a factor $\langle n^2 \rangle = (1 + \tilde{\eta}/2\delta)\langle n \rangle^2$, with each un-paired index contributing simply $\langle n \rangle$. The number of pairings is $\sum_{i=1}^{\ell} \sum_{j=1}^{\ell'} \delta_{s_i s'_j}$, so

$$
\begin{aligned}
2\overline{\Xi}_{s_0 s'_0} &= \sum_{\ell\ell' \geq 0} (-\delta)^{\ell+\ell'} \sum_{s_0 > \cdots > s_\ell} \sum_{s'_0 > \cdots > s'_\ell} D_{s_\ell s'_\ell} \left[1 + \frac{\tilde{\eta}}{2\delta} \right]^{\sum_{i=1}^{\ell} \sum_{j=1}^{\ell'} \delta_{s_i s'_j}} \\
&\quad \times G_{s_0 s_1}, \ldots, G_{s_{\ell-1} s_\ell}\, G_{s'_0 s'_1}, \ldots, G_{s'_{\ell'-1} s'_\ell} \\
&= \sum_{\ell\ell' \geq 0} (-\delta)^{\ell+\ell'} \sum_{s_0 > \cdots > s_\ell} \sum_{s'_0 > \cdots > s'_\ell} D_{s_\ell s'_\ell} \prod_{i=1}^{\ell} \prod_{j=1}^{\ell'} \left[1 + \delta_{s_i s'_j} \frac{\tilde{\eta}}{2\delta} \right] \\
&\quad \times G_{s_0 s_1}, \ldots, G_{s_{\ell-1} s_\ell}\, G_{s'_0 s'_1}, \ldots, G_{s'_{\ell'-1} s'_\ell}.
\end{aligned}
\tag{5.84}
$$

We can finally take the limit $\delta \to 0$, using the substitutions $\delta_{tt'} \to \delta.\delta(t - t')$ and $\sum_s A_{ts} B_{st'} \to \delta^{-1} \int ds\, A(t,s) B(s,t') = (AB)(t,t')$, and find

$$
\begin{aligned}
\overline{\Xi}(s_0, s'_0) &= \frac{1}{2} \sum_{\ell\ell' \geq 0} (-1)^{\ell+\ell'} \int_0^\infty \mathrm{d}s_1, \ldots, \mathrm{d}s_\ell \mathrm{d}s'_1, \ldots, \mathrm{d}s'_\ell \prod_{i=1}^{\ell} \prod_{j=1}^{\ell'} \left[1 + \frac{1}{2} \tilde{\eta} \delta(s_i - s'_j) \right] \\
&\quad \times [1 + C(s_\ell, s'_\ell)] G(s_0, s_1), \ldots, G(s_{\ell-1}, s_\ell) G(s'_0, s'_1), \ldots, G(s'_{\ell'-1}, s'_{\ell'}).
\end{aligned}
\tag{5.85}
$$

Even apart from the usual factor two, we see that after the continuous time limit the volatility matrix (5.85) and the noise covariance matrix (5.55) are still not identical; they differ in whether or not one takes into account time pairings involving s_0 or s'_0.

For the volatility σ, where we already have $s_0 = s'_0$ by definition, the ordering of the integration times imposed by the response functions in equation (5.85) forbids any further pairing of times involving either s_0 or s'_0, so here the differences between equations (5.85) and (5.55) become irrelevant, and we just find

$$
\begin{aligned}
\sigma^2 &= \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathrm{d}t\, \overline{\Xi}(t,t) \\
&= \lim_{\tau \to \infty} \int_0^\tau \frac{\mathrm{d}t}{2\tau} \sum_{\ell\ell' \geq 0} (-1)^{\ell+\ell'} \int_0^\infty \mathrm{d}s_1, \ldots, \mathrm{d}s_\ell \mathrm{d}s'_1, \ldots, \mathrm{d}s'_\ell \\
&\quad \times G(t, s_1) G(s_1, s_2) \ldots G(s_{\ell-1}, s_\ell)\, G(t, s'_1) G(s'_1, s'_2) \ldots G(s'_{\ell'-1}, s'_{\ell'}) \\
&\quad \times [1 + C(s_\ell, s'_{\ell'})] \prod_{i=1}^{\ell} \prod_{j=1}^{\ell'} \left[1 + \frac{1}{2} \tilde{\eta} \delta(s_i - s'_j) \right].
\end{aligned}
\tag{5.86}
$$

5.4.4 Volatility approximation in terms of persistent objects

In ergodic TTI stationary states with finite susceptibility χ we have already seen, when proving the equivalence of batch and on-line dynamics at the level of the persistent order parameter equations, that the various time averages eliminate the learning rate $\tilde{\eta}$ from expressions such as $\overline{\Xi}(\infty)$. We may simply follow the derivation of equation (5.67) and just leave out the forbidden pairings, which gives us directly the (exact) result

$$\overline{\Xi}(\infty) = \frac{1+c}{2(1+\chi)^2}. \tag{5.87}$$

Now let us investigate whether the same happens with expression (5.86) for the volatility. To do so we transform our time variables again, by putting $s_k = \sum_{r=k}^{\ell} t_r$ for all $k = 1, \ldots, \ell$ and $s'_k = \sum_{r=k}^{\ell'} t'_r$ for all $k = 1, \ldots, \ell'$. This allows us to write expression (5.86) as

$$\begin{aligned}
\sigma^2 = & \sum_{\ell\ell' \geq 0} (-1)^{\ell+\ell'} \int_0^{\infty} \mathrm{d}t_0, \ldots, \mathrm{d}t_{\ell-1}\, G(t_0), \ldots, G(t_{\ell-1}) \\
& \times \int_0^{\infty} \mathrm{d}t'_0, \ldots, \mathrm{d}t_{\ell'-1}\, G(t'_0), \ldots, G(t'_{\ell'-1}) \\
& \times \lim_{\tau\to\infty} \int_0^{\tau} \frac{\mathrm{d}t}{2\tau} \int_0^{\infty} \mathrm{d}t_\ell\, \delta\Big[t_\ell - t + \sum_{r=0}^{\ell-1} t_r\Big] \int_0^{\infty} \mathrm{d}t'_{\ell'}\, \delta\Big[t'_{\ell'} - t + \sum_{r=0}^{\ell'-1} t'_r\Big] \\
& \times [1 + C(t_\ell - t'_{\ell'})] \prod_{i=1}^{\ell} \prod_{j=1}^{\ell'} \left[1 + \frac{1}{2}\tilde{\eta}\delta\Big[\sum_{r=i}^{\ell} t_r - \sum_{r=j}^{\ell'} t'_r\Big]\right].
\end{aligned} \tag{5.88}$$

Unlike the evaluation of $\Sigma(\infty)$ in TTI stationary states, here we unfortunately do not find the learning rate dropping out. This can be seen upon using the relations imposed by the δ-distributions in the third line, i.e. $\sum_i t_i = \sum_i t'_i = t$, to transform the arguments of the δ-function and the correlation function in the fourth line

$$\begin{aligned}
\delta\Big[\sum_{r=i}^{\ell} t_r - \sum_{r=j}^{\ell'} t'_r\Big] &\to \delta\Big[\sum_{r<i} t_r - \sum_{r<j} t'_r\Big] \\
C(t_\ell - t'_{\ell'}) &\to C(\sum_{i<\ell} t_i - \sum_{i<\ell'} t_i).
\end{aligned}$$

These two objects have thereby lost their dependences on the times $(t_\ell, t'_{\ell'})$, so that we may apparently write

$$\sigma^2 = \sum_{\ell\ell' \geq 0} (-1)^{\ell+\ell'} \int_0^{\infty} \mathrm{d}t_0, \ldots, \mathrm{d}t_{\ell-1}\, G(t_0), \ldots, G(t_{\ell-1})$$

$$
\begin{aligned}
&\times \int_0^\infty \mathrm{d}t'_0, \ldots, \mathrm{d}t_{\ell'-1}\, G(t'_0), \ldots, G(t'_{\ell'-1}) \\
&\times \prod_{i=1}^{\ell} \prod_{j=1}^{\ell'} \left[1 + \frac{1}{2}\tilde{\eta}\delta\Big[\sum_{r<i} t_r - \sum_{r<j} t'_r\Big]\right] \Big[1 + C(\sum_{i<\ell} t_i - \sum_{i<\ell'} t_i)\Big] \\
&\times \lim_{\tau\to\infty} \int_0^\tau \frac{\mathrm{d}t}{2\tau}\, \theta\Big[t - \sum_{r=0}^{\ell-1} t_r\Big] \theta\Big[t - \sum_{r=0}^{\ell'-1} t'_r\Big] \\
= &\sum_{\ell\ell'\geq 0} (-1)^{\ell+\ell'} \int_0^\infty \mathrm{d}t_0, \ldots, \mathrm{d}t_{\ell-1}\, G(t_0), \ldots, G(t_{\ell-1}) \\
&\times \int_0^\infty \mathrm{d}t'_0, \ldots, \mathrm{d}t_{\ell'-1}\, G(t'_0), \ldots, G(t'_{\ell'-1}) \\
&\times \prod_{i=1}^{\ell} \prod_{j=1}^{\ell'} \left[1 + \frac{1}{2}\tilde{\eta}\delta\Big[\sum_{r<i} t_r - \sum_{r<j} t'_r\Big]\right] \Big[1 + C(\sum_{i<\ell} t_i - \sum_{i<\ell'} t_i)\Big]. \qquad (5.89)
\end{aligned}
$$

This expression is still fully exact, but not easy to use. We see that upon expanding the product that contains $\tilde{\eta}$, each term containing $\tilde{\eta}$ would come with one or more δ-distributions, which lead to further complicated non-vanishing terms.

To proceed to approximations involving persistent order parameters only, we have no choice but to neglect the Poissonnian fluctuations induced by the random selection of histories, and put $\tilde{\eta} \to 0$. Upon returning to equation (5.86) we may now write

$$
\begin{aligned}
\sigma^2 &= \lim_{\tau\to\infty} \int_0^\tau \frac{\mathrm{d}t}{2\tau} \sum_{\ell\ell'\geq 0} (-1)^{\ell+\ell'} \int_0^\infty \mathrm{d}s\mathrm{d}s'\, G^{\ell}(t-s) G^{\ell'}(t-s')\, [1 + C(s-s')] \\
&= \frac{1+c}{2(1+\chi)^2} + \lim_{\tau\to\infty} \int_0^\tau \frac{\mathrm{d}t}{2\tau} \int \mathrm{d}s\mathrm{d}s'\, (\mathbb{1}+G)^{-1}(t-s)\tilde{C}(s-s')(\mathbb{1}+G)^{-1}(s'-t) \\
&= \frac{1+c}{2(1+\chi)^2} + \frac{1}{2} \int \mathrm{d}s\mathrm{d}s'\, (\mathbb{1}+G)^{-1}(s)\tilde{C}(s-s')(\mathbb{1}+G)^{-1}(s'). \qquad (5.90)
\end{aligned}
$$

Here we have defined, as before, the non-persistent correlations as $\tilde{C}(t) = C(t) - c$, and we have introduced the unit operator $\mathbb{1}$ with kernel representation $\mathbb{1}(x, y) = \delta(x - y)$. Expression (5.90) is seen to be the direct on-line MG twin of formula (4.145). However, whereas expression (4.145) was exact for the batch model (where history selection fluctuations are indeed absent), expression (5.90) for the on-line model is an approximation.

As in the batch MG, exact evaluation of equation (5.90) still requires knowledge of our order parameter kernels for short temporal separations. Approximate expressions in terms of persistent order parameters can be constructed along the lines of those in

Chapter 4. For instance, we know that $\tilde{C}(0) = 1 - c$ and $\tilde{C}(\pm\infty) = 0$. A simple approximation would be to assume that the non-persistent correlations $\tilde{C}(t)$ decay very fast, so that $\tilde{C}(t \neq 0)$ is small:

$$\begin{aligned}\sigma^2 &= \frac{1+c}{2(1+\chi)^2} + \frac{1}{2}\int \mathrm{d}s\mathrm{d}s' \left[\delta(s) - G(s) + G^2(s) + \cdots\right] \tilde{C}(s-s') \\ &\quad \times \left[\delta(s') - G(s') + G^2(s') + \cdots\right] \\ &= \frac{1+c}{2(1+\chi)^2} + \frac{1}{2}(1-c) \\ &\quad - \int \mathrm{d}s\, \tilde{C}(s) \Big\{ G(s) - G^2(s) - \frac{1}{2}\int \mathrm{d}s'\, G(s')G(s'-s) + \cdots \Big\}.\end{aligned}$$

Neglecting all integrals over $\tilde{C}$ thus gives:

$$\sigma_A^2 = \frac{1+c}{2(1+\chi)^2} + \frac{1}{2}(1-c). \tag{5.91}$$

From this simple argument we are seen to recover our earlier approximation (4.154) as was obtained in the context of the replica calculation. It will be clear that alternative approximations might also be constructed, along the lines followed in the batch MG.

5.5 Impact of truncating the Kramers–Moyal expansion

A further advantage in having an exact dynamical theory is that it allows one to obtain a better understanding in retrospect of the strengths and weaknesses of past approximations. For instance, in Chapter 3 a theory was constructed for the on-line MG with fake market histories, on the basis of deterministic microscopic laws that are equivalent to truncating the full exact Kramers–Moyal expansion (2.30) and (2.36) after the 0th order $\mathcal{O}(\tilde{\eta}^0)$ (in effect taking the limit $\tilde{\eta} \to 0$). This theory was still shown to be surprisingly successful, at least in the ergodic regime of the MG. In the more advanced version of the replica approach (which we have not discussed in detail in this book), aimed at incorporating also the microscopic fluctuations, one starts instead with the Fokker–Planck equation[25] (2.39). This is seen to be equivalent to truncating the full exact Kramers-Moyal expansion (2.30) and (2.36) after the first order $\mathcal{O}(\tilde{\eta}^1)$, rather than the 0th order.

It turns out that the present formulation of the exact dynamical solution for the on-line MG allows one to trace precisely the effects of such truncations. The deterministic and Fokker–Planck approximations are both equivalent to carrying out an expansion

[25] In practice further approximations are being made in the replica approach, simplifying the diffusion terms in equation (2.39) to obtain equation (3.86).

of the right-hand side of equation (5.1), up to orders $\mathcal{O}(\tilde{\eta}^0)$ or $\mathcal{O}(\tilde{\eta})$, respectively. These expansions would simply propagate through our previous derivations of the intermediate expressions for the generating functional, affecting only the specific exponent which contains the learning rate $\tilde{\eta}$, all the way to expression (5.32). Here we see that approximating the true process by the deterministic or Fokker–Planck truncations ultimately implies making an expansion of the function $\phi[\{x, w\}, u]$, either up to order $\mathcal{O}(\tilde{\eta}^0)$ or to order $\mathcal{O}(\tilde{\eta})$, respectively. Such expansions are carried out easily, and are found to give the following alternative (approximation) formulae

$$\phi_{\text{det}}[\{x, w\}, u] = \mathrm{e}^{\mathrm{i}\delta \sum_t w_t(u+x_t)}, \tag{5.92}$$

$$\phi_{\text{FP}}[\{x, w\}, u] = \mathrm{e}^{\mathrm{i}\delta \sum_t w_t(u+x_t) - \frac{1}{4}\tilde{\eta}\delta \sum_t w_t^2(u+x_t)^2}. \tag{5.93}$$

By simply working out the averages $\langle \ldots \rangle_{\{n\}}$ given below explicitly, one can confirm easily that these two approximations of the true function $\phi[\{x, w\}, u]$ can again be written in the form (5.33), provided we replace the correct Poissonnian distribution $P_a[\ell]$, with $a = 2\delta/\tilde{\eta}$, by appropriate alternatives

$$\phi[\{x, w\}, u] = \left\langle \exp\left[\frac{1}{2} i\tilde{\eta} \sum_t n_t w_t [u + x_t]\right] \right\rangle_{\{n\}} \tag{5.94}$$

with, respectively

$$\begin{aligned} \text{exact}: &\quad P_a[n] = \tfrac{a^n}{n!}\mathrm{e}^{-a} && \langle n \rangle = a, \quad \langle n^2 \rangle - \langle n \rangle^2 = a \\ \text{F}-\text{P approx}: &\quad P_a[n] = \tfrac{1}{\sqrt{2\pi a}}\mathrm{e}^{-\frac{1}{2a}(n-a)^2} && \langle n \rangle = a, \quad \langle n^2 \rangle - \langle n \rangle^2 = a \\ \text{deterministic approx}: &\quad P_a[n] = \delta[n-a] && \langle n \rangle = a, \quad \langle n^2 \rangle - \langle n \rangle^2 = 0. \end{aligned}$$

Apparently, in the Fokker–Planck approximation one replaces in effect the true Poissonnian distribution, which describes the noise induced by the randomly drawn histories, by a Gaussian distribution with the correct first two moments. In the deterministic approximation one replaces the true distribution by a δ-distribution; here only the first moment (the average) is correct.

Since we have taken care to formulate the remaining derivations following (5.32) in terms of expression (5.94) only, and to express all averages over the numbers $\{n_t\}$ in terms of their moments, it is now a trivial matter to determine the regimes of validity of the deterministic and the Fokker–Planck approximations. We can verify that in all averages over the numbers $\{n_t\}$, which we have performed in working towards the final dynamic order parameter equations, only the first two moments of the distribution $P_a[n]$ have ever been required. From this it follows immediately that upon using the Fokker-Planck truncation (provided with the correct diffusion matrix elements, i.e. with expression (2.39) rather than (3.86)) one would have found the correct order parameter equations. In contrast, the deterministic approximation would have led to correct results only for those quantities that in the full derivation were found to be independent of the learning rate $\tilde{\eta}$. This includes the persistent order parameters in the ergodic regime, and the location of the phase transition, but excludes any dynamics and the phenomenology in the non-ergodic regime.

5.6 An assessment

The main result of the theory described in this chapter is to have put in place the mathematical machinery for the application of generating functional techniques to MG versions with on-line rather than batch dynamics (which are, after all, the versions which were put forward first). These on-line models are mathematically more complicated to analyze, due to the extra randomness in the selection of 'fake histories' in the course of the dynamics, which had been specifically 'defined away' in their batch counterparts in order to simplify the mathematics.

In doing so, we have been able to clarify analytically the similarities and differences between having batch or on-line dynamics in the MG, and to show that the stationary state equations in the ergodic regime of the on-line MG are indeed identical to those of the batch version, including the nature and the location of the phase transition. This equivalence had been suggested quite clearly by simulation data and by the replica calculation of Chapter 3, but had not been proven rigorously. However, the equivalence between batch and on-line dynamics at the level of persistent observables in the stationary state was found to extend neither to the transient dynamics of the macroscopic order parameters, nor to the volatility. Furthermore, at the mathematical level we found that it is not generally true that the noise covariance matrix Σ of the effective single-agent process describing the macroscopic theory of the MG is proportional to the volatility matrix $\overline{\Xi}$ (as had been the case for the batch model).

Our generating functional techniques have also allowed us to arrive at a satisfactory understanding of the potential and limitations of various approximations of the underlying microscopic laws, which had been proposed and implemented for the on-line MG in early studies based on the replica method.

Finally, the present analytical exercise can be regarded as a test run for the analysis in Chapter 8 of the more demanding MG models in which the agents are responding to true rather than fake market history. By definition, a batch version of an MG with true market memory cannot exist, so that the effect of having true versus fake market information cannot ever be studied within the mathematically more convenient arena of batch models.

6. The overall bid distribution

6.1 Bid statistics and time-translation invariance

At this stage we make a small detour. Rather than introducing and analyzing new MG versions, we pause briefly and return to the intriguing simulation data on the overall bid statistics as shown for instance in Figs 1.2 (for the on-line MG with true market history) and 1.11 (for the online MG with 'fake' market history). For MGs with fake market history we know by definition that all 'histories' are sampled uniformly, so that for $S = 2$ we can define the time-dependent asymptotic and disorder-averaged bid distribution as follows

$$P_t(A) = \lim_{N\to\infty} \frac{1}{p} \sum_{\mu=1}^{p} \overline{\Big\langle \delta[A - A^\mu[\mathbf{q}(t), \mathbf{z}(t)]] \Big\rangle}, \tag{6.1}$$

with $p = \alpha N$, with the instantaneous overall bid $A^\mu[\ldots]$ as given by equation (2.23), and with the decision noise definitions (2.12) and (2.13). From equation (6.1) we immediately obtain the long-time overall bid distribution via[26]

$$P(A) = \lim_{\tau\to\infty} \lim_{N\to\infty} \frac{1}{\tau p} \sum_{t=1}^{\tau} \sum_{\mu=1}^{p} \overline{\Big\langle \delta[A - A^\mu[\mathbf{q}(t), \mathbf{z}(t)]] \Big\rangle} = \frac{1}{\tau} \sum_{t=1}^{\tau} P_t(A). \tag{6.2}$$

In the case of continuous time, for the on-line MG, the time summation in (6.2) is replaced by an integral. Apart from the limit $N \to \infty$ and the usual assumption of self-averaging with respect to the disorder, expression (6.2) is indeed the distribution shown for the on-line fake history MG in Fig. 1.11. We have seen that in the non-ergodic (i.e. low α) regime of the conventional MG this distribution $P(A)$ is strongly non-Gaussian, which should be sufficient motivation for us to now try calculating it analytically.

[26] This would have been different for the MG with true market history, where time averaging need not and will not automatically involve uniform sampling of the histories μ.

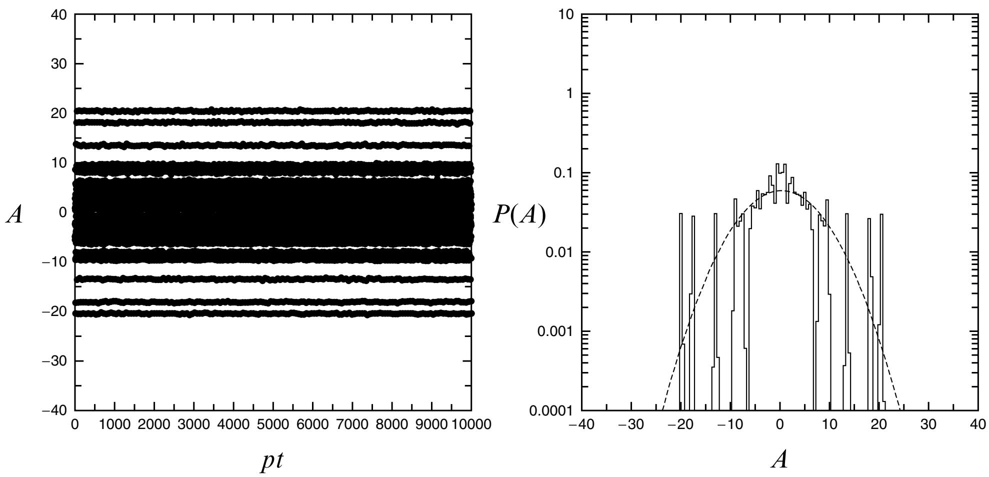


Fig. 6.1 Simulation measurements of the total bids A in the stationary state, observed under conditions identical to those of Figs. 1.2 and 1.11 (again with $N = 4097$, $M = 5$, and $S = 2$, and the same total number of bid evaluations), but now for the batch MG with 'fake' history. The timescale in the left picture gives the total number of bid evaluations at a given time, i.e. pt. The distribution $P(A)$ in the right picture is represented as a histogram, and plotted logarithmically, together with a Gaussian distribution (dashed) of width and average identical to that of the observed $P(A)$.

6.1.1 Statistics in the batch MG with fake history

It turns out that in the previous chapters we have in fact already put in place all the ingredients necessary for calculating $P_t(A)$, and hence also $P(A)$. To see this, let us write the δ-distribution in equation (6.1) in integral representation:

$$P_t(A) = \int \frac{\mathrm{d}\hat{A}}{2\pi}\, \mathrm{e}^{-\mathrm{i}\hat{A}A} \lim_{N\to\infty} \frac{1}{p} \sum_{\mu=1}^{p} \overline{\left\langle \mathrm{e}^{\mathrm{i}\hat{A}A^{\mu}[\mathbf{q}(t),\mathbf{z}(t)]} \right\rangle},$$
$$= \int \frac{\mathrm{d}\hat{A}}{2\pi}\, \mathrm{e}^{-\mathrm{i}\hat{A}A} \lim_{N\to\infty} \frac{1}{p} \sum_{\mu=1}^{p} \overline{Z[\hat{A}\hat{\mathbf{e}}^{\mu}(t)/\sqrt{2}]} \qquad (6.3)$$

Here $\hat{\mathbf{e}}^{\mu}(t)$ denotes the special generating field defined by $\mathrm{e}^{\mu}_{\nu}(t') = \delta_{\mu\nu}\delta_{tt'}$, and $Z[\phi]$ is the generating functional (4.57) which we used earlier to derive an expression for the volatility matrix. We may now put to use explicitly the interpretation of our generating functionals as Fourier transforms, and we will be able to show that for the batch models time-translation invariance automatically implies Gaussian overall bid distributions.

We do not yet know to what extent the overall bid distribution $P(A)$ in the high volatility phase of the batch MG will have the same non-trivial shape as that observed in Figs 1.2 and 1.11 for the on-line models. However, Fig. 6.1, which shows such simulation data for the $S = 2$ batch MG with 'fake' history under otherwise identical conditions, confirms that although this batch distribution $P(A)$ is certainly different from those of figures 1.2 and 1.11 (one clearly observes the expected more regular temporal evolution for the overall bids), the global bids in these data for the batch MG are again non-Gaussian.

For the batch MG we may now call upon our previous results (4.61), (4.62), (4.64), and (4.65). Inserting $\phi = \hat{A}\hat{\mathbf{e}}^{\mu}(t)/\sqrt{2}$ into equation (4.65) gives

$$\Phi = -\alpha \log \det(\mathbf{I} + G^{\dagger}) - \frac{1}{4N}\hat{A}^2[(\mathbf{I}+G)^{-1}D(\mathbf{I}+G^{\dagger})^{-1}]_{tt}. \tag{6.4}$$

Thus we find, with our previous shorthand $\{\mathcal{D}\} = \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}K\mathcal{D}\hat{K}\mathcal{D}L\mathcal{D}\hat{L}$ and with the normalization $Z[\mathbf{0}] = 1$, that according to equations (4.61), (4.62), and (4.64):

$$\lim_{N\to\infty} \frac{1}{p}\sum_{\mu=1}^{p} \overline{Z[\hat{A}\hat{\mathbf{e}}^{\mu}(t)/\sqrt{2}]} = \lim_{N\to\infty} \frac{\int\{\mathcal{D}\}\mathrm{e}^{N[\Psi+\Phi+\Omega]-\frac{1}{4}\hat{A}^2[(\mathbf{I}+G)^{-1}D(\mathbf{I}+G^{\dagger})^{-1}]_{tt}}}{\int\{\mathcal{D}\}\mathrm{e}^{N[\Psi+\Phi+\Omega]}}$$
$$= \mathrm{e}^{-\frac{1}{4}\hat{A}^2[(\mathbf{I}+G)^{-1}D(\mathbf{I}+G^{\dagger})^{-1}]_{tt}} = \mathrm{e}^{-\frac{1}{2}\hat{A}^2\Xi_{tt}}. \tag{6.5}$$

Here the kernels G and D are to be evaluated at the saddle-point of $\Psi+\Phi+\Omega$, i.e. given via the solution of the effective single trader equation for the batch MG. The last step in equation (6.5) could be taken via our earlier identification (4.70) of the volatility matrix in the batch MG. Insertion of equation (6.5) into equation (6.3) subsequently gives the desired (exact) expressions for the instantaneous and long-time overall bid distributions

$$P_t(A) = \int \frac{\mathrm{d}\hat{A}}{2\pi}\, \mathrm{e}^{-\mathrm{i}\hat{A}A-\frac{1}{2}\hat{A}^2\Xi_{tt}} = \frac{\mathrm{e}^{-\frac{1}{2}A^2/\Xi_{tt}}}{\sqrt{2\pi\Xi_{tt}}}, \tag{6.6}$$

$$P(A) = \int \frac{\mathrm{d}\Xi\, W(\Xi)}{\sqrt{2\pi\Xi}} \mathrm{e}^{-\frac{1}{2}A^2/\Xi}, \tag{6.7}$$

$$W(\Xi) = \lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=1}^{\tau} \delta[\Xi - \Xi_{tt}]. \tag{6.8}$$

From these results it follows immediately that in time-translation invariant (TTI) states, where $\Xi_{tt} = \sigma^2$ for all t, we will simply have $W(\Xi) = \delta[\Xi - \sigma^2]$ and hence

$$\text{TTI}: \quad P_t(A) = P(A) = \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}A^2/\sigma^2}. \tag{6.9}$$

Thus, either the observed non-Gaussian overall bid distribution in the high volatility state of the non-ergodic phase in the batch MG reflects finite size effects[27], or the

[27] This implies that it will vanish if we increase the system size N while keeping α fixed.

high volatility state in the batch MG is one without time-translation invariance (such that Ξ_{tt} continues to change with time). Since it seems upon simply observing the bid evolution in Fig. 6.1 that the instantaneous fluctuation strength Ξ_{tt} is for all practical purposes stationary, we are led to the suggestion that at least for the batch MG the observed non-Gaussian shape of $P(A)$ might well be a finite size effect.

6.1.2 Statistics in the on-line MG with fake history

In the on-line MG of Chapter 5 we can calculate $P(A)$ in more or less the same way. Before having taken the continuum time limit $\delta \to 0$, the definition of the generating functional $Z[\boldsymbol{\phi}]$ is identical to that of the batch MG, and the relevant equations are now (5.73), (5.74), (5.76), and (5.78). Inserting $\boldsymbol{\phi} = \hat{A}\hat{\mathbf{e}}^{\mu}(t)/\sqrt{2}$ into (5.78) gives us

$$\Phi = \frac{1}{N} \log \left\langle \mathrm{e}^{-\frac{1}{4}\hat{A}^2 \sum_{tt'} [(\mathbf{1}+GE)^{-1}D(\mathbf{1}+EG^{\dagger})^{-1}]_{tt}} \right\rangle_{\{n\}} \\ -\alpha \log \langle \mathrm{Det}[\mathbf{1} + G^{\dagger}E] \rangle_{\{n\}}. \tag{6.10}$$

Thus we find, with our previous short-hand $\{\mathcal{D}\} = \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}K\mathcal{D}\hat{K}\mathcal{D}L\mathcal{D}\hat{L}$ and with the normalization $Z[\mathbf{0}] = 1$, that

$$\lim_{N\to\infty} \frac{1}{p} \sum_{\mu=1}^{p} \overline{Z[\hat{A}\hat{\mathbf{e}}^{\mu}(t)/\sqrt{2}]} \\ = \lim_{N\to\infty} \lim_{\boldsymbol{\phi}\to\mathbf{0}} \left\langle \frac{\int \{\mathcal{D}\} \mathrm{e}^{N[\Psi+\Phi+\Omega] - \frac{1}{4}\hat{A}^2 [(\mathbf{1}+GE)^{-1}D(\mathbf{1}+EG^{\dagger})^{-1}]_{tt}}}{\int \{\mathcal{D}\} \mathrm{e}^{N[\Psi+\Phi+\Omega]}} \right\rangle_{\{n\}} \\ = \left\langle \mathrm{e}^{-\frac{1}{4}\hat{A}^2 [(\mathbf{1}+GE)^{-1}D(\mathbf{1}+EG^{\dagger})^{-1}]_{tt}} \right\rangle_{\{n\}}. \tag{6.11}$$

Again the various kernels are to be evaluated at the relevant physical saddle-point, i.e. to be solved via the effective single-agent process for the on-line MG. We note that, in contrast to the batch MG, here the instantaneous distributions $P_t(A)$ need *not* be Gaussian. They involve the diagonal entries of the in principle random object

$$\Xi[\{n\}]_{tt} = \lim_{\delta\to 0} \frac{1}{2} [(\mathbf{1} + GE)^{-1} D(\mathbf{1} + EG^{\dagger})^{-1}]_{tt} \\ = \lim_{\delta\to 0} \frac{1}{2} \sum_{\ell\ell'\geq 0} (-\delta)^{\ell+\ell'} \sum_{t>s_1>\cdots>s_\ell} \sum_{t>s'_1>\cdots>s'_{\ell'}} \prod_{i=1}^{\ell} \left[\frac{n_{s_i}}{\langle n\rangle}\right] \prod_{j=1}^{\ell'} \left[\frac{n_{s'_i}}{\langle n\rangle}\right] \\ \times [1 + C_{s_\ell s'_{\ell'}}]\ G_{ts_1} G_{s_1 s_2}, \ldots, G_{s_{\ell-1}s_\ell}\ G_{ts'_1} G_{s'_1 s'_2}, \ldots, G_{s'_{\ell'-1}s'_{\ell'}}. \tag{6.12}$$

We now find alternative expressions for $P_t(A)$ and $P(A)$, which differ from those of the batch MG in more than just having integrals instead of time summations

$$P_t(A) = \lim_{\delta \to 0} \Big\langle \int \frac{\mathrm{d}\hat{A}}{2\pi}\, \mathrm{e}^{-\mathrm{i}\hat{A}A - \frac{1}{2}\hat{A}^2 \overline{\Xi}[\{n\}]_{tt}} \Big\rangle_{\{n\}} = \lim_{\delta \to 0} \Big\langle \frac{\mathrm{e}^{-\frac{1}{2}A^2/\overline{\Xi}[\{n\}]_{tt}}}{\sqrt{2\pi\, \overline{\Xi}[\{n\}]_{tt}}} \Big\rangle_{\{n\}}, \quad (6.13)$$

$$P(A) = \int \frac{\mathrm{d}\Xi\, W(\Xi)}{\sqrt{2\pi\Xi}} \mathrm{e}^{-\frac{1}{2}A^2/\Xi}, \quad (6.14)$$

$$W(\Xi) = \lim_{\delta \to 0} \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{t=1}^{\tau} \Big\langle \delta\Big[\Xi - \overline{\Xi}[\{n\}]_{tt}\Big] \Big\rangle_{\{n\}}. \quad (6.15)$$

Calculating $W(\Xi)$ is thus not a trivial matter; it is similar to but even worse than calculating the volatility (an object which we have as yet only been able to calculate approximately). At most one will be able to calculate the first few moments of $W(\Xi)$. However, even without further calculations it is clear that the random nature of the quantities $\overline{\Xi}[\{n\}]_{tt}$ might well prevent $W(\Xi)$ from being a δ-peak, and consequently $P(A)$ from becoming Gaussian even in TTI states. Hence, in contrast to the batch MG, for the on-line MG it might well be the case that the non-trivial shape of $P(A)$ is not a finite size effect, but will persist for arbitrary (large) system sizes.

6.2 Finite size and finite time effects

6.2.1 Size dependence of bid statistics

We have thus been led to the investigation via numerical simulations of the bid distributions $P(A)$, for different system sizes N, in both the batch and the on-line MG. Upon repeating the simulation experiments of Figs 1.2, 1.11, and 6.1 for larger system sizes N (while keeping the ratio $\alpha = 2^M/N$ fixed), focusing on the high volatility state in the non-ergodic regime, we find the predictions of the theory regarding the potential effects of finite system sizes on the overall bid distribution confirmed experimentally. Typical data are shown in Fig. 6.2. In the batch MG, increasing the system size leads to the non-Gaussian 'outlier' data being consistently moved into (and absorbed by) the central part of the distribution $P(A)$, which thereby approaches a Gaussian shape as predicted. In contrast, a different mechanism is found to be at work in the two on-line models, whether with true or with fake market histories: here the non-Gaussian outliers move in the opposite direction, away from the centre, and retain statistical significance, by their very presence preventing the bid distribution from becoming Gaussian. Upon extrapolating this trend, one can even appreciate the danger of these outlier peaks continuing their relocation to large values of A as M is increased further, which raises the prospect of $P(A)$ converging at most point-wise to a limiting function in the limit $N \to \infty$.

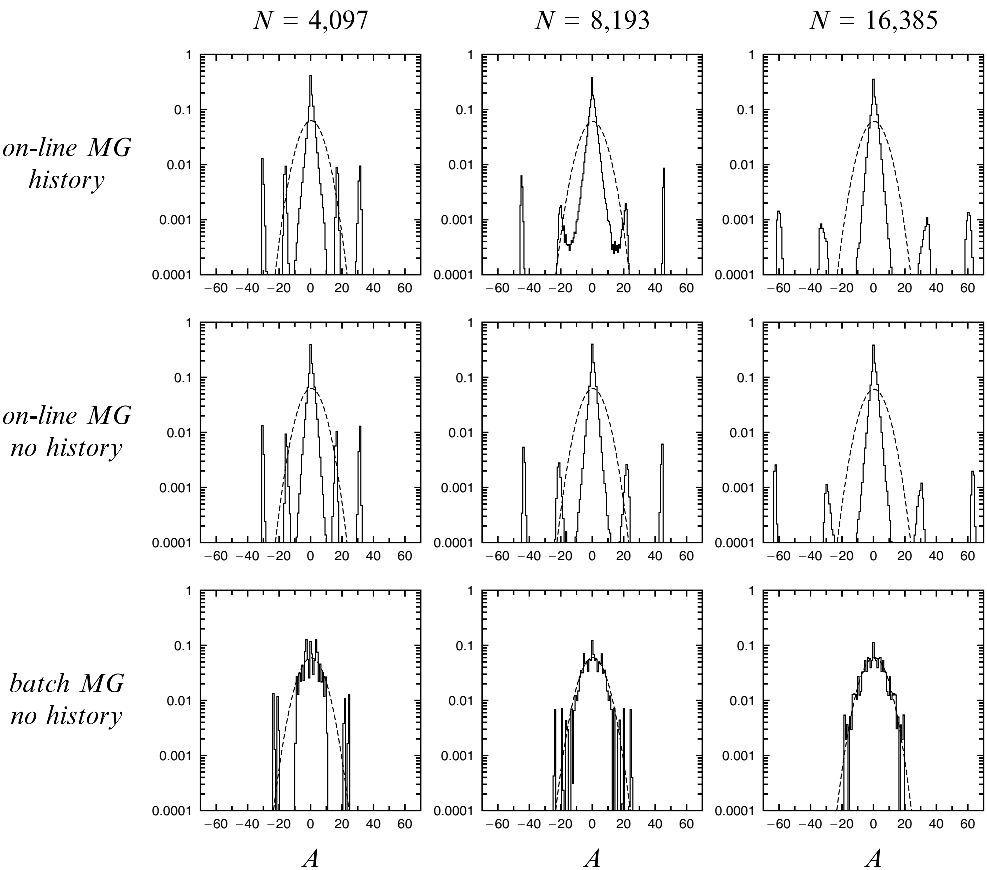


Fig. 6.2 The overall bid distributions $P(A)$ as typically observed in numerical simulations following 'tabula rasa' initial conditions of the three MG types analyzed so far (represented as histograms, and plotted logarithmically), together with Gaussian distributions (dashed) of width and average identical to those of the observed $P(A)$. All cases refer to the same ratio $\alpha = p/N = 1/128$, but with different system sizes N.

A natural final question is whether for on-line dynamics perhaps at least the 'central' part of the distribution $P(A)$ might be Gaussian. To find out whether this is the case one can simply repeat the numerical analysis that led to Fig. 6.2 for the on-line MG versions, but now while leaving out all values for the bid A which belong to the outliers of $P(A)$. The result is shown in Fig. 6.3. We seem to be forced to conclude that in both on-line MG versions (with true and with fake market history), in the non-ergodic phase[28] also the central part of $P(A)$ is not of a Gaussian shape (the simulations seem

[28] In contrast, in the ergodic large α phase the distribution $P(A)$ is always found to become Gaussian for sufficiently large N, in both batch and on-line MG versions.

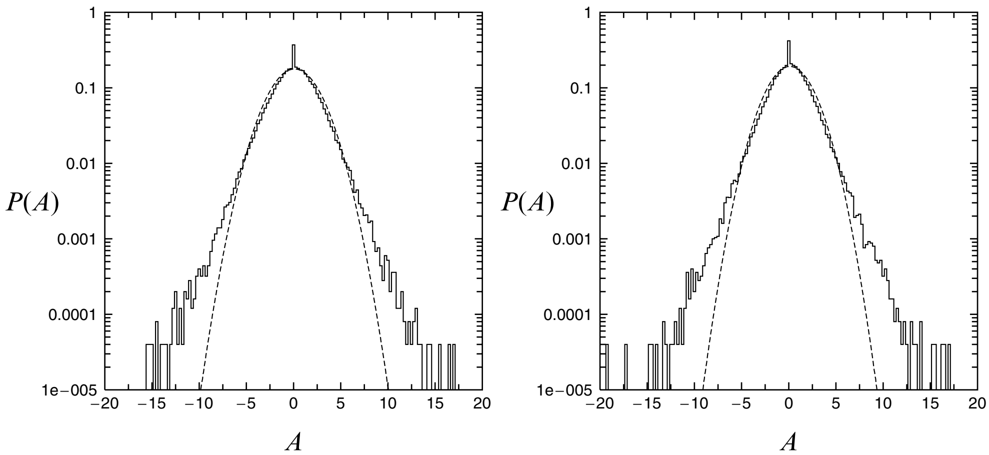


Fig. 6.3 Simulation data for the total bids A in the on-line MG with true (left) and fake (right) histories, for $N = 16{,}385$ and $\alpha = 1/128$, as in the top right and middle right panels of the previous figure. Here we compare the data to Gaussian distributions (dashed) of width and average identical to those of the observed $P(A)$ when taking into consideration only the *central* part of the distribution (i.e. when leaving out all data with $|A| > 20$).

to point to exponential tails), which underlines even more the differences between the on-line and batch MG versions in terms of their overall bid statistics.

6.2.2 Dependence on equilibration times

Let us next turn to the question of whether the observed non-Gaussian bid statistics in the high volatility states of the on-line MG versions could perhaps be transient phenomena. This can be investigated by repetition of the previous simulation experiments, but now following different equilibration periods (or ‘waiting times’), see Fig. 6.4. Within the framework of our analyses the allowed timescales of individual iterations as labelled by $\ell = 0, 1, 2, 3, \ldots$ for on-line models are required to scale with the system size N as $\ell = \mathcal{O}(N)$. Figure 6.4 indicates that on such timescales the bid statistics are not affected by increasing the equilibration times, so that we may conclude that the non-Gaussian shape of $P(A)$ is not a transient phenomenon in an infinite system. However, as soon as one chooses timescales of order $\ell = \mathcal{O}(N^2)$ (which are not described by the present theories), one will increasingly tend to observe transitions where the system moves from the high volatility solution with non-Gaussian bid statistics as in Fig. 6.4 to the low volatility solutions with Gaussian bid statistics of the type shown in Fig. 1.8. On such longer timescales, ergodicity is apparently being restored, in favour of the low volatility states.

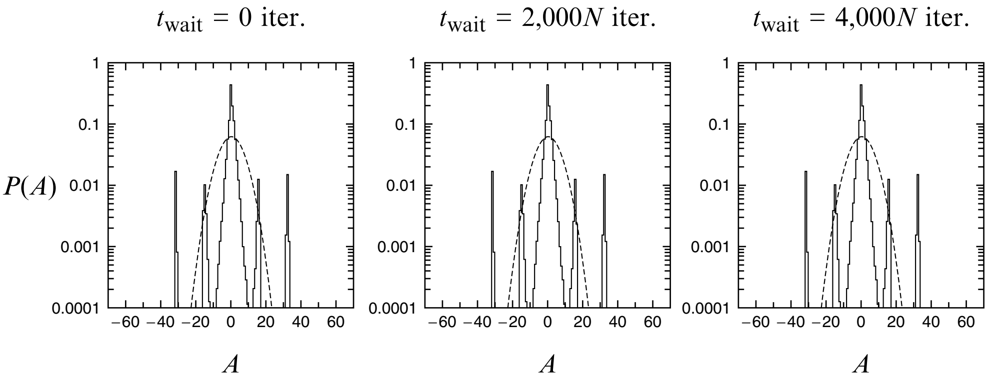


Fig. 6.4 The overall bid distributions $P(A)$ as typically observed in numerical simulations following 'tabula rasa' initial conditions of the on-line MG with fake market histories (represented as histograms, and plotted logarithmically), together with Gaussian distributions (dashed) of width and average identical to those of the observed $P(A)$. All graphs refer to $N = 4{,}097$ and $\alpha = 1/128$. In all three simulations the measurements were taken during a period of $2{,}000N$ iterations, but following different waiting times.

In carrying out simulation experiments to determine overall bid statistics one has to be aware that occasionally the system can also find itself trapped temporarily in meta-stable macroscopic states (resulting in the top middle graph in Fig. 6.2). Given sufficient time, of order $\mathcal{O}(N)$ individual iterations, the system will at some point escape from such metastable states to the final stationary state (whether in the high or the low volatility branch). An example of this happening in a simulation is shown in Fig. 6.5.

6.3 Measuring bid predictability

In all MG models encountered so far (and in most models still to be discussed) one finds that, in TTI states without anomalous response, only one specific aspect of the overall bid statistics can be expressed without approximations in terms of persistent order parameters only, namely

$$\overline{\Xi}(\infty) = \frac{1+c}{2(1+\chi)^2}. \tag{6.16}$$

Let us inspect the meaning and relevance of this quantity in somewhat more detail. Given absence of anomalous response (so that we are in the ergodic regime, and no phase transitions have been encountered yet), one may expect the limits $N \to \infty$ and $\lim_{\tau\to\infty} \tau^{-1} \sum_{t\leq\tau}$ to commute, and also to find for arbitrary single-time observables

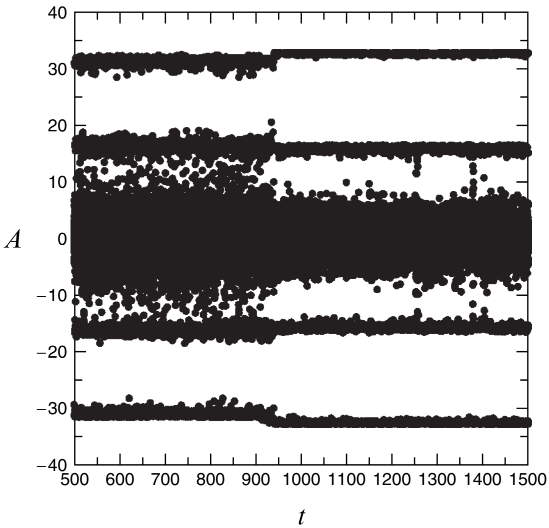


Fig. 6.5 Example of the on-line MG with 'fake' histories being temporarily 'trapped' in (and subsequently escaping from) a metastable macroscopic state (here the relation between the natural $\mathcal{O}(1)$ time t and the number of underlying individual iterations ℓ is given by $\ell = 2Nt$).

that $\lim_{\tau\to\infty} \tau^{-1} \sum_{t\leq\tau} x(t) = \lim_{t\to\infty}\langle x(t)\rangle$. For the batch MG we may therefore infer from equation (4.146) that

$$\overline{\Xi}(\infty) = \lim_{N\to\infty} \frac{1}{p} \sum_{\mu=1}^{p} \overline{\left\langle \Big[\lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t\leq\tau} A^\mu[\mathbf{q}(t), \mathbf{z}(t)] \Big]^2 \right\rangle}$$

$$= \lim_{N\to\infty} \lim_{t\to\infty} \frac{1}{p} \sum_{\mu=1}^{p} \overline{\langle A^\mu[\mathbf{q}(t), \mathbf{z}(t)]\rangle^2}. \tag{6.17}$$

In combination with the two definitions

$$A^\mu[\mathbf{q}(t), \mathbf{z}(t)] = \Omega_\mu + \frac{1}{\sqrt{N}} \sum_j \xi_j^\mu \sigma[q_j(t), z_j(t)], \quad \sigma[q] = \int \mathrm{d}z\, P(z)\sigma[q, z]$$

it now follows that

$$\overline{\Xi}(\infty) = \lim_{N\to\infty} \lim_{t\to\infty} \frac{1}{N} \overline{\langle H(\boldsymbol{\sigma}[\mathbf{q}(t)])\rangle} = E, \tag{6.18}$$

where $H(\boldsymbol{\sigma}[\mathbf{q}]))$ is the Lyapunov function (3.10) which formed the cornerstone of the pseudo-equilibrium replica analysis in Chapter 3. This establishes a further link between the two approaches[29]. The above identities are also found to hold in the case of the on-line MG with fake market histories.

We note that the quantity $\langle A^{\mu}[\mathbf{q}(t), \mathbf{z}(t)]\rangle$ in equation (6.17) can be interpreted as the expectation value of the overall bid at time t, conditional on observing (pseudo) history μ at time t. Therefore, as soon as $E = \overline{\overline{\Xi}}(\infty) > 0$ (E can never be negative, by definition) there is at least a non-zero fraction of histories with a measurable bias in the value of the overall bid following upon observation of these histories. This allows for better than random prediction of the overall bid at time t once we are given the history $\mu(t)$. Conversely, at least in MGs with fake market histories and for Gaussian distributed bids we may safely conclude that having $\overline{\overline{\Xi}}(\infty) = 0$ implies that there is no way to predict the sign of $A(t)$ upon knowing $\mu(t)$. This has prompted the use of E as a measure of *predictability* in MGs with fake market histories.[30] It should be emphasized that the precise amount of information revealed about the bid $A(t)$ by knowledge of the history $\mu(t)$ is not given by E; the former would require the calculation of more sophisticated information theoretical measures, such as the disorder-averaged mutual information between histories and subsequent bids.

A final conclusion following from the interpretation of $\overline{\overline{\Xi}}(\infty)$ as a measure of predictability is that the $\chi = \infty$ phase transition in the batch and on-line MGs (with fake histories) can be interpreted alternatively as separating a regime where there is *information* in the history data $\{\mu(t)\}$ (the regime $\alpha > \alpha_{\rm c}(T)$) from one where there is no such information (the regime $\alpha < \alpha_{\rm c}(T)$). This would suggest that if, for a fixed value of p, agents were allowed to join the game or withdraw from it on the basis of the potential for overall bid prediction, one would expect the number N to increase for as long as $\alpha > \alpha_{\rm c}$ and to decrease whenever $\alpha < \alpha_{\rm c}$. Since $\alpha = p/N$, one is then led to the intriguing prediction that the MG would automatically be driven by such a mechanism to operate permanently at its critical point.

[29] One has to be somewhat careful in comparing the stationary state formulae of the replica and generating functional formalisms. We have established so far that we may identify the trio $\{\overline{\overline{\Xi}}(\infty), c, \phi\}$ with $\{E, q, \phi\}$ of Chapter 3. The definition of the quantity χ in the two formalisms, however, is seen to differ by a factor α. This could obviously have been repaired, but would have made comparison of the various formulae with those found in literature more difficult.

[30] For the MG with true market histories, matters are, of course, much more subtle. Here, there may and generally will be further information in the history strings at previous times, i.e. even if $A(t)$ cannot be predicted on the basis of $\mu(t)$ alone, this does not imply that it cannot be predicted on the basis of, say, the pair $\{\mu(t), \mu(t-1)\}$.

7. MG versions with new types of phase transitions

The generating functional method allows us for $N \to \infty$ to reduce the problem of solving the original coupled and disordered N-agent process to that of solving a stochastic and non-Markovian 'effective single-agent' equation, without disorder. So far our strategy in analyzing this single-agent equation has been to first look for ergodic states, described by TTI stationary solutions, with finite susceptibility χ. We assumed that, if such a solution exists (which was found to be true in each of the MG versions studied so far, for sufficiently large α), given time, the system will forget its initial conditions and approach this particular stationary state. Numerical simulations have so far confirmed this assumed scenario. The predictions for the persistent observables of the MG, as well as for the location of the point where this particular solution should break down (marked by a divergence of χ) and the resulting phase diagrams, agreed perfectly with our experimental observations.

In this chapter we inspect two examples of MG models where the above standard picture breaks down, and novel transitions can take place *before* the point where χ diverges. In the first model, where agents are allowed to correct their microscopic equations for their own impact on the global market bid, we will find a regime in the phase diagram where, although a TTI state again exists and the susceptibility is indeed finite, this state is not approached by the system on finite timescales.[31] This gives rise to a new type of phase transition, characterized by the onset of long-term memory, to a new regime where ergodicity is itself not yet violated (χ is finite, so there is still only one ergodic sector) but where the system cannot fully shed off its initial conditions. In the second model, the so-called spherical MG, there are no problems with the emergence of long-term memory, but again we find a novel phase transition which is not marked by a divergence of the susceptibility. This model has been designed such that its effective single-trader equation is solvable, in all regions of the phase diagram,

[31] Since in our generating functional analysis we always have to take the limit $t \to \infty$ *after* the limit $N \to \infty$, we must conclude that here in a finite system the TTI state is approached at most on timescales which grow and diverge with N, if at all.

which allows us to probe much further into the behaviour of the system.[32] Here the new transition separates an oscillating high α phase from a frozen phase, without the phase boundary involving a divergence of χ.

7.1 The batch MG with self-impact correction

7.1.1 Definition of self-impact correction

We will now define a generalization of the batch MG, in which the standard phase transition phenomenology, which has been found so far in our previous MG versions, will be quite different. To appreciate the idea behind this generalization, we first return to equations (4.2) and (4.3) for the simple batch MG with 'fake' market memory (the same arguments apply to other MG versions), viz.

$$q_i(t+1) = q_i(t) - \frac{2}{\sqrt{N}} \sum_{\mu=1}^{p} \xi_i^\mu A^\mu[\mathbf{q}(t), \mathbf{z}(t)], \tag{7.1}$$

$$A^\mu[\mathbf{q}, \mathbf{z}] = \Omega_\mu + \frac{1}{\sqrt{N}} \sum_i \sigma[q_i, z_i] \xi_i^\mu. \tag{7.2}$$

The logic behind these equations was for each agent i to reward/punish his profitable/loss-making strategies, by comparing each strategy's proposed bid at each stage t in response to 'data' entry μ with the actual global bid $A^\mu[\mathbf{q}(t), \mathbf{z}(t)]$. However, each agent i in the game will always be able to separate the overall bid $A^\mu[\mathbf{q}(t), \mathbf{z}(t)]$ into an external contribution (the combined contributions of all agents $j \neq i$), and the contribution reflecting his own trading decision (which for agent i is obviously known). Let us make this distinction of contributions explicit, by writing the 'corrected' bid as

$$A_i^\mu[\mathbf{q}, \mathbf{z}] = \frac{1}{\sqrt{N}} \sum_{j \neq i} \Big[\omega_j^\mu + \sigma[q_j, z_j] \xi_j^\mu \Big]. \tag{7.3}$$

Each agent i is able to separate $A^\mu[\mathbf{q}, \mathbf{z}] = A_i^\mu[\mathbf{q}, \mathbf{z}] + N^{-\frac{1}{2}} \{\omega_i^\mu + \sigma[q_i, z_i] \xi_i^\mu\}$. The batch MG equations (7.1) and (7.2) can thereby be written in the following form, using the two properties $\xi_i^\mu \omega_i^\mu = \frac{1}{4}(R_\mu^{i1} - R_\mu^{i2})(R_\mu^{i1} + R_\mu^{i2}) = 0$ (since $R_\mu^{ia} = \pm 1$) and $p^{-1} \sum_{\mu \leq p} (\xi_i^\mu)^2 = \frac{1}{2} + \mathcal{O}(p^{-\frac{1}{2}})$:

$$q_i(t+1) = q_i(t) - \frac{2}{\sqrt{N}} \sum_{\mu=1}^{p} \xi_i^\mu A_i^\mu[\mathbf{q}(t), \mathbf{z}(t)] - \alpha \sigma[q_i(t), z_i(t)] \frac{2}{p} \sum_{\mu=1}^{p} (\xi_i^\mu)^2$$

[32] The need for constructing a model for which the effective single-trader equation is fully solvable was prompted by the observation that the analysis of these effective equations is presently the main underdeveloped part of the generating functional analysis research programme.

$$= q_i(t) - \frac{2}{\sqrt{N}} \sum_{\mu=1}^{p} \xi_i^\mu A_i^\mu[\mathbf{q}(t), \mathbf{z}(t)] - \alpha\sigma[q_i(t), z_i(t)] + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right). \quad (7.4)$$

We see that the term reflecting an agent's own impact on the market, which is the third term in equation (7.4), carries no useful information at all, but acts only as a consistent destabilizing negative feedback force in the system, as a result of $\text{sgn}[\langle\sigma[q_i(t), z_i(t)]\rangle_{\mathbf{z}}] = -\text{sgn}[q_i(t)]$.

We now define the batch MG with market impact correction by the equations which one obtains by allowing for a (partial) removal of the undesirable term in the above equations, i.e. we put

$$q_i(t+1) = q_i(t) - \frac{2}{\sqrt{N}} \sum_{\mu=1}^{p} \xi_i^\mu \Big\{ A^\mu[\mathbf{q}(t), \mathbf{z}(t)] - \frac{\rho}{\sqrt{N}} \Big[\omega_i^\mu + \sigma[q_i(t), z_i(t)]\xi_i^\mu\Big] \Big\}. \quad (7.5)$$

Equivalently, at least for look-up table entries of the type $R_\mu^{ia} = \pm 1$ (with equal probabilities and distributed independently), we may write this rule as

$$q_i(t+1) = q_i(t) - \frac{2}{\sqrt{N}} \sum_{\mu=1}^{p} \xi_i^\mu A^\mu[\mathbf{q}(t), \mathbf{z}(t)] + \alpha\rho\sigma[q_i(t), z_i(t)] + \mathcal{O}\left(\frac{1}{\sqrt{N}}\right). \quad (7.6)$$

We see that for $\rho = 0$ we return to the model of Chapter 4, whereas for $\rho = 1$ the agents correct fully for their own impact on the market. We will keep $\rho \in [0, 1]$ as a control parameter of our model, in addition to α and the noise level T.

7.1.2 Generating functional analysis

The generating functional analysis of the process (7.6), equipped with the usual external fields $\{\theta_i(t)\}$, requires only minor modifications of that developed for the ordinary batch MG in Chapter 4. We define as before

$$Z[\boldsymbol{\psi}] = \langle e^{i \sum_{t\geq 0} \sum_i \psi_i(t)\sigma[q_i(t), z_i(t)]} \rangle \quad (7.7)$$

but now the path average refers to the process corresponding to the following transition probability density

$$W_t(\mathbf{q}|\mathbf{q}') = \int \frac{d\hat{\mathbf{q}}}{(2\pi)^N} \Big\langle e^{i \sum_i \hat{q}_i [q_i - q_i' - \theta_i(t) + \frac{2}{\sqrt{N}} \sum_\mu \xi_i^\mu A^\mu[\mathbf{q}', \mathbf{z}] - \frac{2\rho}{N} \sum_\mu (\xi_i^\mu)^2 \sigma[q_i', z_i]]} \Big\rangle_{\mathbf{z}}. \quad (7.8)$$

This implies that we can go from the generating functional in Chapter 4 to the one describing the present model with (partial) self-impact correction by simply substituting a factor $\exp[-\frac{2i\rho}{N}\sum_{i\mu}(\xi_i^\mu)^2\hat{q}_i(t)s_i(t)]$ into expression (4.17). This factor propagates through the disorder average, which thereby becomes only slightly more involved, and ultimately instructs us to subtract the term $i\alpha\rho\sum_{it}\hat{q}_i(t)s_i(t)+\mathcal{O}(\sqrt{N})$ from the exponent of the disorder average (4.18). Finally one may absorb this additional term into the function Ω (4.26), which thereby becomes

$$\Omega = \frac{1}{N}\sum_i \log\Big\langle \int \mathcal{D}q\mathcal{D}\hat{q}\, p_0(q(0)) \times \mathrm{e}^{\mathrm{i}\sum_t[\hat{q}(t)[q(t+1)-q(t)-\theta_i(t)-\alpha\rho\sigma[q(t),z(t)]]+\psi_i(t)\sigma[q(t),z(t)]]-\mathrm{i}\sum_{tt'}\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')} \times \langle \mathrm{e}^{-\mathrm{i}\sum_{tt'}[\hat{C}_{tt'}\sigma[q(t),z(t)]\sigma[q(t'),z(t')]+\hat{K}_{tt'}\sigma[q(t),z(t)]\hat{q}(t')]}\Big\rangle_{\mathbf{z}}. \tag{7.9}$$

This is the only change in the previous generating functional of Chapter 4 that will be induced by the new self-impact correction term. All of the more complicated derivations that were carried out previously to simplify the saddle-point equations and the effective single-agent problem for the original batch MG, continue to apply unaltered. One can hence immediately proceed to the checkout and write down the final effective single-agent equation for the present MG version with (partial) market impact correction

$$q(t+1) = q(t)+\theta(t)-\alpha\sum_{t'\le t}[(\mathbf{1}+G)^{-1}-\rho\mathbf{1}]_{tt'}\sigma[q(t'),z(t')]+\sqrt{\alpha}\,\eta(t) \tag{7.10}$$

in which $z(t)$ refers to the decision noise, and $\eta(t)$ is the usual disorder-induced effective Gaussian noise, with zero mean and with correlations $\langle\eta(t)\eta(t')\rangle = \Sigma_{tt'}$ which are given by

$$\Sigma = (\mathbf{1}+G)^{-1}D(\mathbf{1}+G^\dagger)^{-1}, \tag{7.11}$$

where $D_{tt'} = 1 + C_{tt'}$. The dynamic order parameters $\{C, G\}$ are as always to be solved self-consistently from the effective single-agent process (7.10), via

$$C_{tt'} = \langle\sigma[q(t),z(t)]\sigma[q(t'),z(t')]\rangle_\star, \quad G_{tt'} = \frac{\partial}{\partial\theta(t')}\langle\sigma[q(t),z(t)]\rangle_\star. \tag{7.12}$$

There are no changes either to the calculation of the volatility matrix of the MG, which continues to be given by $\Xi_{tt'} = \frac{1}{2}\Sigma_{tt'}$. We see in equation (7.10) that for $\rho = 0$ (i.e. no self-impact correction) we do indeed return to the previous equation (4.55). For $\rho = 1$ (i.e. full self-impact correction), on the other hand, it follows from the identity $(\mathbf{1}+G)^{-1}-\mathbf{1} = \sum_{\ell>0}(-1)^\ell G^\ell$ that the instantaneous contribution to the self-interaction term in the effective single-agent process (7.10) is precisely being cancelled.

7.1.3 TTI stationary states

Let us next inspect the changes that will be caused by the self-impact correction to the previous TTI stationary state solutions of the batch MG, i.e. the solutions where $C_{tt'} = C(t-t')$ and $G_{tt'} = G(t-t')$ for all (t,t'), and with finite susceptibility $\chi = \int \mathrm{d}t\ G(t)$. As before we transform the original stochastic single-agent variable $q(t)$ to a new variable $\tilde{q}(t) = q(t)/t$. Here this gives in the absence of perturbation fields

$$\tilde{q}(t) = \frac{q(0)}{t} + \frac{\sqrt{\alpha}}{t}\sum_{t'=0}^{t-1}\eta(t') - \frac{\alpha}{t}\sum_{t'=0}^{t-1}\sum_{s\geq 0}[(\mathbb{1}+G)^{-1} - \rho\mathbb{1}]_{t's}\sigma[s\tilde{q}(s), z(s)]. \tag{7.13}$$

Upon taking the limit $t \to \infty$ in this equation, assuming the limit $\tilde{q} = \lim_{t\to\infty}\tilde{q}(t)$ to exist and χ to be finite, we obtain

$$\tilde{q} = \sqrt{\alpha}\,\overline{\eta} - \frac{\alpha\overline{\sigma}(1-\rho-\rho\chi)}{1+\chi}, \tag{7.14}$$

with $\overline{\eta} = \lim_{\tau\to\infty}\tau^{-1}\sum_{t\leq\tau}\eta(t)$, with $\overline{\sigma} = \lim_{\tau\to\infty}\tau^{-1}\sum_{t\leq\tau}\sigma[\tilde{q}t]$, and with $\sigma[q] = \int \mathrm{d}z\ P(z)\sigma[q,z]$. As before this equation allows for 'fickle solutions' and 'frozen' solutions, with conditions for existence which are expressed in terms of the realization of the zero-average Gaussian variable $\overline{\eta}$. However, here we have to be much more careful, since we can no longer we sure beforehand of the signs of $1-\rho-\rho\chi$ and $1+\chi$. For instance, just putting $\rho = 1$ in equation (7.14) already shows that sign changes may well occur.

We therefore first study the case where the signs of the relevant quantities do remain as they were for the ordinary batch MG, i.e. $(1-\rho-\rho\chi)/(1+\chi) > 0$. Physically this means, according to equation (7.14), that the effective stationary retarded self-interaction of the single agent continues to act as a negative feedback force. Given that $\overline{\sigma} = \text{sgn}[\tilde{q}].\sigma[\infty]$ for frozen solutions, and $\tilde{q} = 0$ for fickle ones, we now obtain

$$|\overline{\eta}| \leq \frac{\sigma[\infty]\sqrt{\alpha}(1-\rho-\rho\chi)}{1+\chi}: \quad \text{'fickle' solution,} \quad \overline{\sigma} = \frac{(1+\chi)\overline{\eta}}{\sqrt{\alpha}(1-\rho-\rho\chi)}, \tag{7.15}$$

$$|\overline{\eta}| > \frac{\sigma[\infty]\sqrt{\alpha}(1-\rho-\rho\chi)}{1+\chi}: \quad \text{'frozen' solution,} \quad \overline{\sigma} = \sigma[\infty]\text{sgn}[\overline{\eta}]. \tag{7.16}$$

What remains is to calculate, via the above classification of solutions, expressions for the variance of $\overline{\eta}$, for $c = \lim_{t\to\infty} C(t)$, for the fraction of frozen agents ϕ, and for the susceptibility χ. Since our formula for the volatility matrix in the present model is identical to that in Chapter 4, we continue to find the old expression

$$\langle\overline{\eta}^2\rangle = \lim_{\tau\to\infty}\frac{1}{\tau^2}\sum_{tt'=0}^{\tau}[(\mathbf{1}+G)^{-1}D(\mathbf{1}+G^{\dagger})^{-1}]_{tt'} = \frac{1+c}{(1+\chi)^2}. \tag{7.17}$$

In order to proceed it will be convenient to define the following auxiliary variable

$$v = \frac{\sqrt{\alpha}|1-\rho-\rho\chi|\sigma[\infty]}{\sqrt{2(1+c)}} \geq 0. \tag{7.18}$$

This definition, together with equation (7.17), allows us to compactify equations (7.15) and (7.16). Upon writing $\overline{\eta} = \langle\overline{\eta}^2\rangle^{\frac{1}{2}}z$, so that z is a unit-variance and zero-average Gaussian variable, we find

$$|z| \leq v\sqrt{2}: \quad \text{'fickle' solution,} \quad \overline{\sigma} = \sigma[\infty].(z/v\sqrt{2}), \tag{7.19}$$

$$|z| > v\sqrt{2}: \quad \text{'frozen' solution,} \quad \overline{\sigma} = \sigma[\infty].\text{sgn}[z]. \tag{7.20}$$

We can now calculate the trio $\{c, \phi, \chi\}$, much as before

$$\begin{aligned} c &= \sigma^2[\infty]\int Dz\,\left\{\theta[v\sqrt{2}-|z|].\frac{z^2}{2v^2}+\theta[|z|-v\sqrt{2}]\right\} \\ &= \sigma^2[\infty]\left\{1+\frac{1-2v^2}{2v^2}\text{Erf}[v]-\frac{1}{v\sqrt{\pi}}\text{e}^{-v^2}\right\}. \end{aligned} \tag{7.21}$$

$$\phi = \int Dz\;\theta[|z|-v\sqrt{2}] = 1-\text{Erf}[v], \tag{7.22}$$

$$\begin{aligned} \chi &= \frac{1}{\sqrt{\alpha}}\langle\frac{\partial}{\partial\overline{\eta}}\overline{\sigma}\rangle_{\star} = \frac{1+\chi}{\alpha(1-\rho-\rho\chi)}\int Dz\;\theta[v\sqrt{2}-|z|] \\ &= \frac{1+\chi}{\alpha(1-\rho-\rho\chi)}\text{Erf}[v]. \end{aligned} \tag{7.23}$$

It follows from equation (7.23) that $\chi \geq 0$. We may now use equation (7.18) and some simple rearrangements to write expression (7.23) alternatively as

$$\chi = \frac{\sigma[\infty](1-\phi)}{v\sqrt{2\alpha(1+c)}-\sigma[\infty](1-\phi)}. \tag{7.24}$$

Insertion of this equation into equation (7.18) gives us a fixed-point equation for v, in which we no longer find χ but only the control parameters $\{\alpha, \rho, T\}$ via the explicit and relatively simple equations (7.21) and (7.22):

$$v = \frac{\sqrt{\alpha}\,\sigma[\infty]}{\sqrt{2(1+c)}}\left|\frac{(1-\rho)v-(\sigma[\infty](1-\phi))/(\sqrt{2\alpha(1+c)})}{v-(\sigma[\infty](1-\phi))/(\sqrt{2\alpha(1+c)})}\right|. \tag{7.25}$$

For $\rho \to 0$ (no impact correction) we recover from equations (7.24) and (7.25) the old equations for the batch MG, including the susceptibility $\chi = (1-\phi)/(\alpha-1+\phi)$.

For $\rho = 1$ (full impact correction), on the other hand, we see that the present solution, based on the simple premise $(1 - \rho - \rho\chi)/(1 + \chi) > 0$, simply cannot exist. The reason is clear: for $\rho = 1$ our premise demands that $\chi/(1+\chi) < 0$, whereas we know from (7.23) that $\chi > 0$.

We thus observe that the apparently harmless and minor modification to our previous batch MG equations, which results from allowing the agents to no longer operate as 'price takers' but instead to correct their equations for their own (weak) impact on the overall bid, has surprisingly serious implications. For any finite $\rho < 1$ we always recover in the large α regime from the above equations a self-consistent solution, with $\phi < 1$ and with negative effective feedback in the single-trader equation (as in the ordinary MG versions), and with asymptotically $\lim_{\alpha\to\infty} v^{-1} = \lim_{\alpha\to\infty} \chi = \lim_{\alpha\to\infty} c = \lim_{\alpha\to\infty} \phi = 0$. As we lower α, however, this solution must break down when $v = 0$. At that point we move into a fully frozen state, characterized by

$$\chi = \rho^{-1} - 1, \quad \phi = 1, \quad c = \sigma^2[\infty]. \tag{7.26}$$

Expansion of our formulae for small values of v, first of equations (7.21) and (7.22) and subsequently of the right-hand side of equation (7.25), will tell exactly when this breakdown will occur

$$\text{RHS} = \frac{\sqrt{\alpha}\,\sigma[\infty]}{\sqrt{2(1+\sigma^2[\infty])}} \left| \frac{1 - \rho - (\sqrt{2}\,\sigma[\infty])/(\sqrt{\alpha\pi(1+\sigma^2[\infty])})}{1 - (\sqrt{2}\,\sigma[\infty])/(\sqrt{\alpha\pi(1+\sigma^2[\infty])})} \right| + \mathcal{O}(v). \tag{7.27}$$

We switch from a state with $\phi < 1$ into the state (7.26) at the point where this particular expression becomes zero, i.e. when α has been lowered to the critical value

$$\alpha_{\text{froz}} = \frac{2\sigma^2[\infty]}{\pi(1-\rho)^2(1+\sigma^2[\infty])}. \tag{7.28}$$

One can easily show that for $\rho \to 0$ (no impact correction) this critical value (7.28) is lower than the critical value marking the old $\chi = \infty$ transition in the batch MG, i.e. we will simply never get there. For $\rho \to 1$ (full impact correction), however, we find $\alpha_{\text{froz}} \to \infty$, i.e. we will find the fully frozen stationary state (7.26) for *all* α and $T < \infty$. At intermediate values $0 < \rho < 1$, the system is predicted to freeze at the point (7.26), without ever experiencing a diverging susceptibility χ. In this sense, both values $\rho = 0$ (no self-impact correction) and $\rho = 1$ (full self-impact correction) are special cases.

Finally we have to inspect the possibility of solutions of our stationary state equations which correspond to $(1 - \rho - \rho\chi)/(1 + \chi) < 0$, i.e. positive effective feedback. Here matters are no longer as clear-cut as with negative feedback. In particular, we can no longer infer the solution type (fickle versus frozen) from the persistent noise $\overline{\eta}$ only, since now

$$|\overline{\eta}| > \frac{\sqrt{\alpha}|1-\rho-\rho\chi|\sigma[\infty]}{|1+\chi|} \quad \text{'frozen' solution,} \quad \overline{\sigma} = \sigma[\infty].\text{sgn}[\overline{\eta}], \quad (7.29)$$

$$|\overline{\eta}| < \frac{\sqrt{\alpha}|1-\rho-\rho\chi|\sigma[\infty]}{|1+\chi|} \quad \text{'frozen' and 'fickle' solutions exist} \quad (7.30)$$

As one could have expected from the presence of positive feedback, remanence effects now allow the system to be in any of the potential states for sufficiently small values of $|\overline{\eta}|$. However, at the point where the change of feedback sign actually occurs as we descent from the large α regime, i.e. at (7.28), we have $1-\rho-\rho\chi = 0$, so that also the above two conditions (7.29) and (7.30) are consistent with having the fully frozen state (7.26) for all $\alpha < \alpha_{\text{froz}}$.

7.1.4 Experimental evidence for violation of TTI

If we solve the above equations (7.21), (7.22), (7.25), (7.26), and (7.28) for our persistent order parameters numerically,[33] and compare the resulting predictions for c and ϕ with the results of carrying out numerical simulations, we find surprising results. Firstly, we observe that the prominent dependence of observables such as c and ϕ on the initial conditions, as was observed in the low α regime of the ordinary batch MG (i.e. for $\rho = 0$ in the above equations), is being reduced drastically as soon as $\rho > 0$. For instance, in Fig. 7.1, we show simulation data for initial conditions of the type $|q_i(0)| = \Delta$ for all i (for the choices $\Delta = 0$ or $\Delta = 10$), together with the predictions extracted from numerical solution of our order parameter equations, for each choice of the self-impact parameter $\rho \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. Only for $\rho = 0$ do we see the familiar differences between the $\Delta = 0$ and the $\Delta = 10$ curves, at small values of α. Exactly as predicted by the theory, we also observe that the $\rho = 1$ simulations indeed confirm that for full self-impact correction, both the observables c and ϕ are independent of α (within the limits of numerical simulation accuracy). Thus, in the case of full self-impact correction, the system indeed evolves into a fully frozen state for any α. However, we also note that for $0 < \rho < 1$ there are serious deviations between theory and simulations, for intermediate values of α. Since we know from our theory that the susceptibility remains finite for $\rho > 0$, these deviations cannot be explained by the present equations. At higher values of α there appears to be excellent agreement between theory and simulations, so our suspicion must be that, although self-consistent TTI solutions are found to exist in the present model, for intermediate values of α and for $\rho > 0$ the system is not able to evolve towards such states.

[33] We note that, as with the previous MG versions, additive decision noise has no influence at all on the ergodic and TTI solutions of our equations.

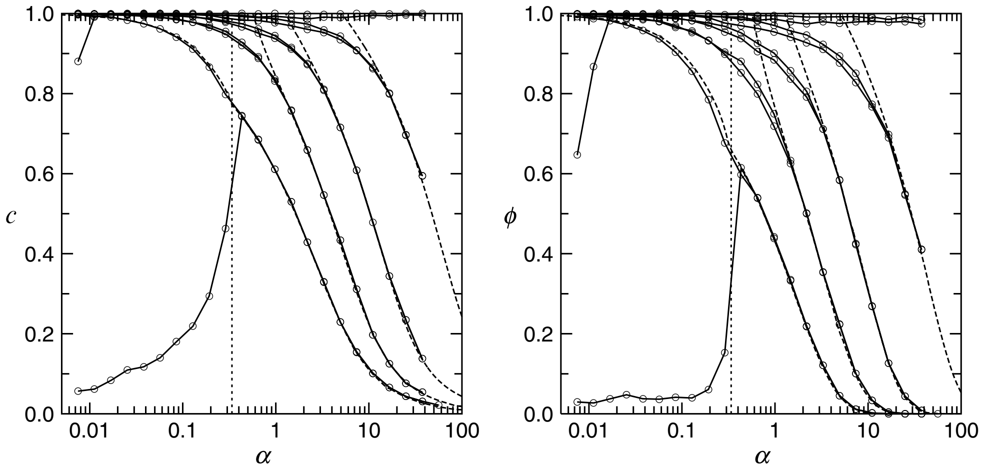


Fig. 7.1 The predicted persistent correlations c (dashed curve, left figure) and the predicted fraction ϕ of frozen agents (dashed curve, right figure), together with simulation data (connected markers), for the batch MG without decision noise but with self-impact correction. Self-impact correction factors: $\rho \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ (lower to upper curves in the high α regime). Initial conditions: $|q_i(0)| = \varDelta$ for all i, data are shown for $\varDelta \in \{0, 10\}$. The vertical dotted line marks the critical value $\alpha_c \approx 0.33740$ for the $\chi \to \infty$ transition (for $\rho = 0$ only; for $\rho > 0$ there is no such transition).

7.1.5 The memory onset transition

The resolution and explanation of the above apparent differences between theory and (simulation) experiment is found to lie in the occurrence of a new transition preceding the $\chi \to \infty$ one, towards a state where the so-called weak long-term memory (WLTM) property is violated

$$\text{WLTM}: \quad (\forall t' \geq 0): \quad \lim_{t\to\infty} G_{tt'} = 0. \tag{7.31}$$

Since violation of WLTM would, for instance, immediately imply $\chi = \infty$ in TTI states, WLTM can be regarded as a more basic property than the combination of TTI and finite susceptibility. In the present model, the physical mechanism responsible for the system not achieving TTI is thought to be the following. During the transients of the dynamics, small perturbations still have the potential of turning otherwise fickle agents into frozen ones, and vice versa. Although ultimately the frozen agents will no longer feel any further perturbations due to the divergence of their strategy preference values $q_i(t)$, they may well forever be frozen into microscopic states which continue to reflect the early perturbations.[34] Thus, after the transients have died out, the TTI

[34] Put differently: the system continues to memorize the perturbations applied in the transient stage of the process, which have left their fingerprints by having influenced the selection of one particular runaway solution (i.e. which of the N agents will have strategy valuations diverging linearly with time) from a range of possibilities.

part of the response function reflects the continuing response of the 'fickle' agents to ongoing perturbations, whereas any contribution to the response function $G_{tt'}$ of the 'frozen' agents (if present), will depend only on the perturbation time t'.

In order to see mathematically how solutions without WLTM may bifurcate from the previously studied TTI ones, let us make an ansatz based on the above physical picture, and separate the response function $G_{tt'}$ explicitly into a TTI part $\tilde{G}$ and a further small contribution $\epsilon\hat{G}$ without TTI but which depends only on the time t' at which the perturbation was applied (i.e. not on the time t at which the effect of this perturbation is being measured)

$$G_{tt'} = \tilde{G}(t-t') + \epsilon\hat{G}(t') \tag{7.32}$$

with $0 < \epsilon \ll 1$. Within our picture, the fact that 'late' perturbations can no longer exert an influence on the frozen agents translates into $\lim_{t'\to\infty}\hat{G}(t') = 0$. The susceptibility χ is still being defined in terms of the TTI part of the response, i.e. $\chi = \sum_{t\geq 0}\tilde{G}(t)$. The weak non-TTI contribution to the response can be calculated from our stationary state equations, based on equation (7.14) and the time average $\overline{\sigma} = \lim_{\tau\to\infty}\tau^{-1}\sum_{t\leq\tau}\sigma[q(t)]$, using the following property (which relies on $\chi < \infty$):

$$\begin{aligned}\frac{\partial}{\partial\theta(t')}\langle\overline{\sigma}\rangle_\star &= \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=t'+1}^{t'+\tau}\frac{\partial}{\partial\theta(t')}\langle\sigma[q(t)]\rangle_\star = \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=t'+1}^{t'+\tau}G_{tt'}\\ &= \epsilon\hat{G}(t') + \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=1}^{\tau}\tilde{G}(t) = \epsilon\hat{G}(t').\end{aligned} \tag{7.33}$$

We next try to analyze the effect of weak WLTM violations of the type (7.32) on our stationary state equations. With the ansatz (7.32) we may expand the retarded self-interaction kernel $(\mathbb{1}+G)^{-1}$ for small $\hat{G}$, and find

$$\begin{aligned}(\mathbb{1}+G)^{-1} &= \sum_{n\geq 0}(-1)^n(\tilde{G}+\epsilon\hat{G})^n\\ &= (\mathbb{1}+\tilde{G})^{-1} + \epsilon\sum_{n>0}(-1)^n\sum_{m=0}^{n-1}\tilde{G}^m\hat{G}\tilde{G}^{n-1-m} + \mathcal{O}(\epsilon^2).\end{aligned} \tag{7.34}$$

In order to use equation (7.33), we introduce a one-off perturbation $\theta(t')$ into the effective single-agent process at time t'. Being non-persistent, this perturbation cannot affect the TTI response, nor will it show up as an explicit contribution in $\tilde{q} = \lim_{t\to\infty}\tilde{q}(t)$ since for $t > t'$:

$$\tilde{q}(t) = \frac{q(0)}{t} + \frac{\theta(t')}{t} + \frac{\sqrt{\alpha}}{t}\sum_{t'=0}^{t-1}\eta(t') - \frac{\alpha}{t}\sum_{t'=0}^{t-1}\sum_{s\geq 0}[(\mathbb{1}+G)^{-1} - \rho\mathbb{1}]_{t's}\sigma[s\tilde{q}(s), z(s)]. \tag{7.35}$$

Upon taking the limit $t \to \infty$ in this equation, assuming the limit $\tilde{q} = \lim_{t\to\infty} \tilde{q}(t)$ to exist and χ to be finite and inserting the expansion (7.34), we obtain

$$\tilde{q} = \sqrt{\alpha}\,\overline{\eta} - \frac{\alpha\overline{\sigma}(1-\rho-\rho\chi)}{1+\chi} - \alpha\epsilon\Delta + \mathcal{O}(\epsilon^2)$$

$$\Delta = \sum_{n>0}(-1)^n \sum_{m=0}^{n-1} \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \sum_{s\geq 0} [\tilde{G}^m \hat{G} \tilde{G}^{n-1-m}]_{ts} \sigma[s\tilde{q}(s), z(s)]. \quad (7.36)$$

Within the relevant orders in ϵ, only the term Δ can still depend on the perturbation $\theta(t')$, via the variables $\sigma[s\tilde{q}(s), z(s)]$. Compared with the previously studied case $\epsilon = 0$, the effect of the introduction of the weak non-TTI response on our expressions for $\tilde{q}$ can be captured by the substitution $\overline{\eta} \to \overline{\eta} - \epsilon\sqrt{\alpha}\Delta + \mathcal{O}(\epsilon^2)$. Hence we find our previous equations (7.15) and (7.16) for $\overline{\sigma}$ being modified to

$$|\overline{\eta} - \epsilon\sqrt{\alpha}\Delta| \leq \frac{\sigma[\infty]\sqrt{\alpha}(1-\rho-\rho\chi)}{1+\chi}: \quad \text{'fickle'}, \quad \overline{\sigma} = \frac{(1+\chi)(\overline{\eta} - \epsilon\sqrt{\alpha}\Delta)}{\sqrt{\alpha}(1-\rho-\rho\chi)},$$

$$|\overline{\eta} - \epsilon\sqrt{\alpha}\Delta| > \frac{\sigma[\infty]\sqrt{\alpha}(1-\rho-\rho\chi)}{1+\chi}: \quad \text{'frozen'}, \quad \overline{\sigma} = \sigma[\infty]\text{sgn}[\overline{\eta} - \epsilon\sqrt{\alpha}\Delta].$$

We may write this more compactly upon introducing the saturation function

$$\text{sat}[z] = \begin{cases} z & \text{if} \quad |z| < 1 \\ \text{sgn}[z] & \text{if} \quad |z| > 1. \end{cases}$$

This definition allows us to state

$$\overline{\sigma} = \sigma[\infty].\text{sat}\Big[\frac{(1+\chi)(\overline{\eta} - \epsilon\sqrt{\alpha}\Delta)}{\sqrt{\alpha}(1-\rho-\rho\chi)\sigma[\infty]}\Big]. \quad (7.37)$$

We are now in a position to evaluate our equation (7.33) for $\hat{G}$, by combining equation (7.37) with definition (7.36):

$$\begin{aligned}
\hat{G}(t') &= -\frac{1+\chi}{1-\rho-\rho\chi}\Big\langle \text{sat}'\Big[\frac{(1+\chi)\overline{\eta}}{\sqrt{\alpha}(1-\rho-\rho\chi)\sigma[\infty]}\Big]\frac{\partial\Delta}{\partial\theta(t')}\Big\rangle_\star + \mathcal{O}(\epsilon) \\
&= -\frac{1+\chi}{1-\rho-\rho\chi}\Big\langle \frac{\partial\Delta}{\partial\theta(t')}\theta\Big[\frac{\sigma[\infty]\sqrt{\alpha}(1-\rho-\rho\chi)}{1+\chi} - |\overline{\eta}|\Big]\Big\rangle_\star + \mathcal{O}(\epsilon) \\
&= -\frac{1+\chi}{1-\rho-\rho\chi}\sum_{n>0}(-1)^n \sum_{m=0}^{n-1} \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau-1}\sum_{s\geq 0}[\tilde{G}^m\hat{G}\tilde{G}^{n-1-m}]_{ts} \\
&\quad \times \Big\langle \theta\Big[\frac{\sigma[\infty]\sqrt{\alpha}(1-\rho-\rho\chi)}{1+\chi} - |\overline{\eta}|\Big]\frac{\partial}{\partial\theta(t')}\sigma[s\tilde{q}(s), z(s)]\Big\rangle_\star + \mathcal{O}(\epsilon).
\end{aligned}$$

The $\mathcal{O}(\epsilon^0)$ term in the last line is seen to be the response of the fickle agents at time s to a perturbation at time t. Within our ansatz (7.32), the response due to fickle agents is exactly the TTI part of the response function, so that the last line is simply equal to $\tilde{G}(s,t')$. This implies that in the limit $\epsilon \to 0$ one has

$$\hat{G}(t') = -\frac{1+\chi}{1-\rho-\rho\chi}\sum_{n>0}(-1)^n \sum_{m=0}^{n-1} \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau-1}[\tilde{G}^m\hat{G}\tilde{G}^{n-m}]_{tt'}. \quad (7.38)$$

We see that this eigenvalue problem can be written in the standard form $\hat{G}(t) = \sum_{t'} M_{tt'}\hat{G}(t')$, where the kernel M can be simplified (using $\chi < \infty$) to

$$\begin{aligned} M_{tt'} &= \frac{1+\chi}{1-\rho-\rho\chi}\sum_{n>0}(-1)^{n-1}\sum_{m=0}^{n-1}\tilde{G}^{n-m}(t'-t)\sum_{s\geq 0}\tilde{G}^m(s)\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t''=t'+s}^{\tau-1}1 \\ &= \frac{1+\chi}{1-\rho-\rho\chi}\sum_{n>0}(-1)^{n-1}\sum_{m=0}^{n-1}\chi^m\tilde{G}^{n-m}(t'-t). \end{aligned} \quad (7.39)$$

Clearly, M is itself TTI, and even a relatively simple algebraic function of the kernels $\tilde{G}^\dagger$ which can be re-summed explicitly

$$\begin{aligned} M^\dagger &= \frac{1+\chi}{1-\rho-\rho\chi}\sum_{n>0}(-1)^{n-1}\sum_{m=0}^{n-1}\chi^m\tilde{G}^{n-m} \\ &= \frac{(1+\chi)\tilde{G}}{1-\rho-\rho\chi}[\mathbf{1}-\tilde{G}/\chi]^{-1}\sum_{n\geq 0}(-1)^n\chi^n[\mathbf{1}-(\tilde{G}/\chi)^{n+1}] \\ &= \frac{(1+\chi)\tilde{G}}{1-\rho-\rho\chi}[\chi\mathbf{1}-\tilde{G}]^{-1}\left[\frac{\chi\mathbf{1}}{1+\chi}-\tilde{G}(\mathbf{1}+\tilde{G})^{-1}\right] \\ &= \frac{\tilde{G}(\mathbf{1}+\tilde{G})^{-1}}{1-\rho-\rho\chi}. \end{aligned} \quad (7.40)$$

We conclude that bifurcations of the type (7.32) correspond to eigenfunctions of the kernel $\tilde{G}^\dagger$, which (as a direct a result of TTI) are simply the Fourier modes $\mathrm{e}^{\mathrm{i}\omega t}$. More specifically, upon insertion we find the following bifurcation condition

$$\hat{G}(t) = \mathrm{e}^{\mathrm{i}\omega t}, \quad 1 = \rho\Big[1+\sum_t \tilde{G}(t)\mathrm{e}^{\mathrm{i}\omega t}\Big]^2. \quad (7.41)$$

The first such bifurcation occurs when $\sum_t \tilde{G}(t)\mathrm{e}^{\mathrm{i}\omega t}$ is real-valued and maximal. If we assume that $\tilde{G}$ does not oscillate (being in the high α regime) this immediately leads us to $\omega = 0$. Thus a transition of the type (7.32), to be called onset of memory (MO), takes place when the following condition is met

$$\text{MO transition}: \qquad \rho = (1+\chi)^{-2}. \quad (7.42)$$

The corresponding non-TTI kernel is of the form $\hat{G}(t) = 1$ (for $t \geq 0$, since causality of $\tilde{G}$ projects out the negative times in the kernel M). This implies that the dominant non-TTI response, if it occurs, is indeed the response to perturbations which happened at the start of the process.

Finally, it is worth mentioning and quite satisfactory that within the pseudo-equilibrium replica analysis as in Chapter 3 one also finds the above instability (7.42): there it marks the point where the assumption of replica symmetry for the saddle-points breaks down.

7.1.6 Explanation of simulations and phase diagram

Clearly, for $\rho = 0$ (i.e. in the batch MG without self-impact correction) there can be no solution of (7.42) before we run into the $\chi = \infty$ transition, since in that case we know that χ increases monotonically as we reduce α, from $\chi = 0$ at $\alpha = \infty$ to its singularity. For $\rho > 0$, however, transitions of the type (7.42) do occur. For $\rho = 1$ (full self-impact correction) it follows from $\lim_{\alpha\to\infty} \chi = 0$ that the MO transition occurs already at $\alpha = \infty$, so there we can *never* achieve TTI for any α. For $0 < \rho < 1$ we know that the system enters a fully frozen state at the point where χ has increased from $\chi = 0$ at $\alpha = \infty$ to the value $\chi = \rho^{-1} - 1$. Hence for $0 < \rho < 1$ the MO transition must necessarily *always* occur at some finite α, before the system reaches the frozen state. If we use equation (7.25) to eliminate ρ, we see after some simple rearrangements of terms that the MO transition condition (7.42) can be also written in the form

$$2v^2(1 + c) = \sigma^2[\infty](1 - \phi). \tag{7.43}$$

Upon inserting into (7.43) our expressions (7.21) and (7.22) for c and ϕ, we then find the remarkable result that, when written in terms of the variable v, the MO transition point is given by exactly the same value for v as the $\chi = \infty$ transition was in the ordinary batch MG. In both cases v is to be solved from

$$\sigma^2[\infty]\left\{\mathrm{Erf}[v] - 1 + \frac{1}{v\sqrt{\pi}}\mathrm{e}^{-v^2}\right\} = 1, \tag{7.44}$$

which is identical to the previous condition (4.133) for the batch MG. Since also the expressions for ϕ and c, in terms of the variable v, are fully independent of ρ, it follows that the MO transition for $\rho > 0$ takes place exactly when the values of ϕ and c are those corresponding to the $\chi = \infty$ transition of the $\rho = 0$ case. Solving (7.44) gives us an expression $v_\mathrm{c}(T)$. Together with (7.43), this can be inserted into Eq. (7.24) for the susceptibility, giving

$$\chi_\mathrm{c}(T) = \frac{\sqrt{\mathrm{Erf}[v_\mathrm{c}(T)]}}{\sqrt{\alpha} - \sqrt{\mathrm{Erf}[v_\mathrm{c}(T)]}}. \tag{7.45}$$

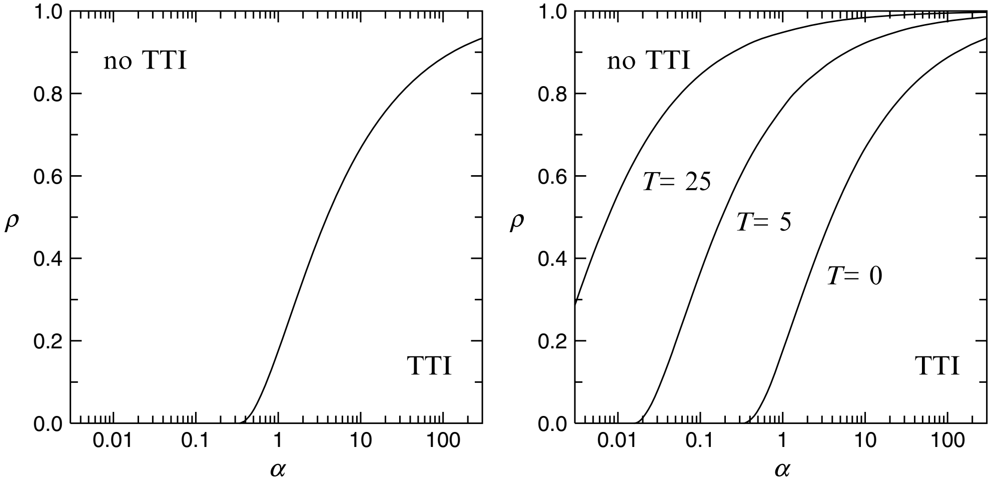


Fig. 7.2 The phase diagrams in the (α, ρ) plane for the batch MG with self-impact correction, separating a phase with TTI solutions (for large α), from one where TTI is not stable (small α) where the system cannot shed off the initial conditions on finite times. Left: absent or additive decision noise (additive noise has no impact on the transition). Right: multiplicative decision noise, with $T \in \{0, 5, 25\}$ (from bottom to top). Only for $\rho = 0$ (no self-impact correction) does the memory onset transition (MO), separating these two phases, coincide with the $\chi \to \infty$ transition, for $\rho > 0$ the TTI property always breaks down before χ diverges.

This result, in turn, allows us to write the MO transition condition (7.42) explicitly in terms of the three control parameters $\{\alpha, T, \rho\}$:

$$\text{MO transition}: \quad \rho_c = \left[1 - \sqrt{\text{Erf}[v_c(T)]/\alpha}\right]^2. \tag{7.46}$$

Since for $\rho = 0$ (the ordinary batch MG) the $\chi = \infty$ transition was given by $\alpha_c(T) = \text{Erf}[v_c(T)]$, we see that the new MO transition touches the previous $\chi = \infty$ transition at $\rho = 0$, for any noise strength T. For additive decision noise there is (as always) no dependence on T, i.e. the MO transition is the same as that found without decision noise. For multiplicative noise, on the other hand, there is a clear dependence on T, with $\lim_{T\to\infty} \rho_c = 1$ for any α. Drawing the transition lines (7.46) in the (α, ρ) plane thus gives rise to the phase diagrams shown in Fig. 7.2.

For all $\rho > 0$ the MO transition is found to be the first transition to occur as we lower α, so we now appear to have found the missing explanation for the earlier deviations between the theoretical curves (based on assuming TTI solutions) and the simulation data in Fig. 7.1. If in the previous figures, showing our simulation data for ϕ and c, we now also draw the line giving the predicted critical values for these observables which mark the MO instability (given according to our above equations by precisely the values achieved for ϕ and c at the old $\chi = \infty$ criticality), we find that the deviations between TTI theory and the simulation data indeed set in at the predicted

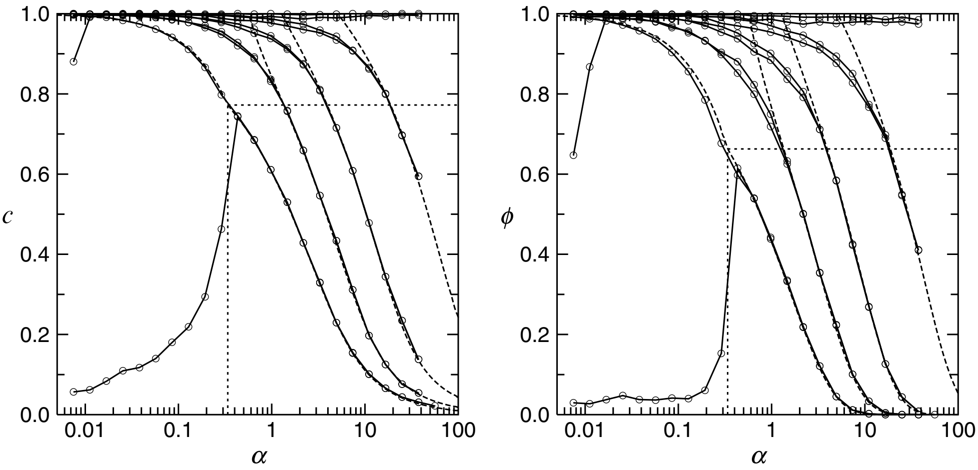


Fig. 7.3 The predicted persistent correlations c (dashed curves, left picture) and the predicted fraction ϕ of frozen agents (dashed curves, right picture), together with simulation data (connected markers), for the batch MG without decision noise but with self-impact correction. Self-impact correction factors: $\rho \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ (lower to upper curves in the high α regime). Initial conditions: $|q_i(0)| = \Delta$ for all i, data are shown for $\Delta \in \{0, 10\}$. The vertical dotted line marks the critical value $\alpha_c \approx 0.33740$ for the $\chi \to \infty$ transition (for $\rho = 0$ only). The horizontal line segments mark the predicted occurrence of the MO instability (with deviations between TTI theory and data predicted above these segments, but not below: $c^\star \approx 0.772$ and $\phi^\star \approx 0.663$).

MO transition point (within the experimental error margins, especially those due to the difficulty in determining the frozen fraction ϕ in the case of multiplicative noise). This is immediately clear from the examples in Fig. 7.3 (no decision noise), Fig. 7.4 (additive decision noise of strength $T = 1$), and Fig. 7.5 (multiplicative decision noise of strength $T = 1$).

One can also test more explicitly the claim made regarding the nature of the MO transition, namely that it marks a point where the system finds itself no longer able to 'forget' its initial conditions, by measuring in numerical simulations the value of the observable C_{t0}. The latter measures the correlations between the microscopic variables $\sigma[q_i(t), z_i(t)]$ at time t and those at time zero. In the TTI regime these must vanish on finite timescales, but this should no longer be the case beyond the MO transition line. Figure 7.6 shows the resulting curves obtained for $\rho = \frac{1}{2}$, $T = 0$, and $\alpha \in \{\frac{5}{4}, \frac{5}{2}, 5, 10\}$; here the system is predicted to forget its initial conditions for $\alpha \in \{5, 10\}$ curves, but not for $\alpha \in \{\frac{5}{4}, \frac{5}{2}\}$. This prediction is borne out quite convincingly.

7.1.7 Effects of self-impact correction on volatility

Possibly the most striking effect of market self-impact correction is exhibited by the volatility σ. The volatility values observed in the numerical simulations of the previous figures (for absent, additive, and multiplicative decision noise, and for impact

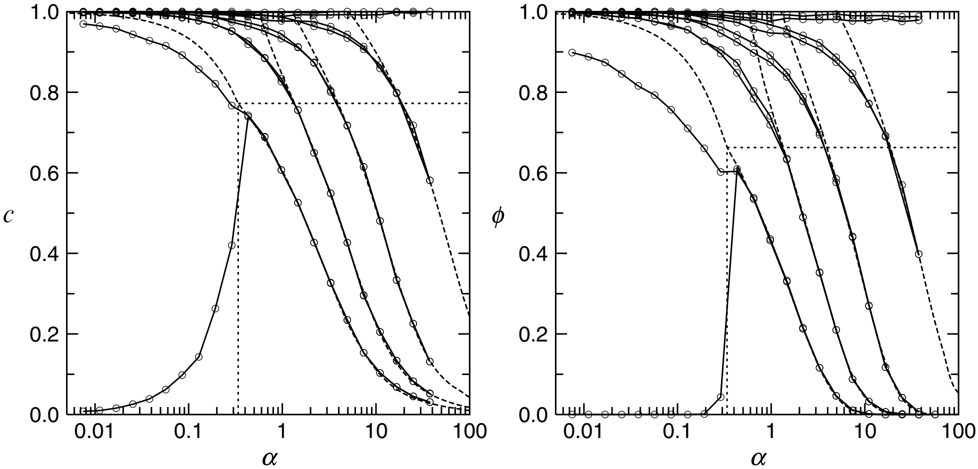


Fig. 7.4 The predicted correlations c (dashed curves, left picture) and the predicted fraction ϕ of frozen agents (dashed curves, right picture), together with simulation data (connected markers), for the batch MG with additive decision noise of strength $T = 1$ and with self-impact correction. Self-impact correction factors: $\rho \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ (lower to upper curves in the high α regime). Initial conditions: $|q_i(0)| = \Delta$ for all i, data are shown for $\Delta \in \{0, 10\}$. The vertical dotted line marks the critical value $\alpha_c \approx 0.33740$ for the $\chi \to \infty$ transition (for $\rho = 0$ only). The horizontal line segments mark the predicted occurrence of the MO instability (with deviations between TTI theory and data predicted above these segments, but not below: $c^\star \approx 0.772$ and $\phi^\star \approx 0.663$).

correction factors $\rho \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$) are shown in figures 7.7 and 7.8. In these figures we also show as dashed lines the approximation (4.155) for the volatility,[35] up to the point where the MO transition (7.42) takes place; similar curves are found for the alternative approximation (4.154).

We see that, apart from a relapse for small values of ρ and very low α, the introduction of even partial market impact correction by all agents in the batch MG not only eliminates the unwanted high volatilities which were seen following ‘tabula rasa’ initialization in the original batch MG, but even reduces the previous low volatilities obtained following biased initialization, by orders of magnitude.[36] Only for multiplicative noise, where it is fundamentally impossible to eliminate all fluctuations (even if

[35] We note that in the derivation of the approximations (4.154) and (4.155) we did not make direct use of the form of the retarded self-interaction in the effective single-agent process, so this formula should be expected to apply also to the present model.

[36] In the ordinary batch MG the approximations (4.154) and (4.155), although generally inaccurate for $\alpha < \alpha_c$, were in the low α regime at least for the low volatility solution qualitatively correct. Here, for $\rho > 0$, it is not at all clear whether and how the TTI solution could be extended sensibly into the non-TTI regime. For instance, the simplest extension, being $\{\phi = 1, c = \sigma[\infty], \chi = \rho^{-1} - 1\}$, would predict below the MO transition point that σ remains constant as α is reduced further. Such behaviour is seen only for multiplicative noise.

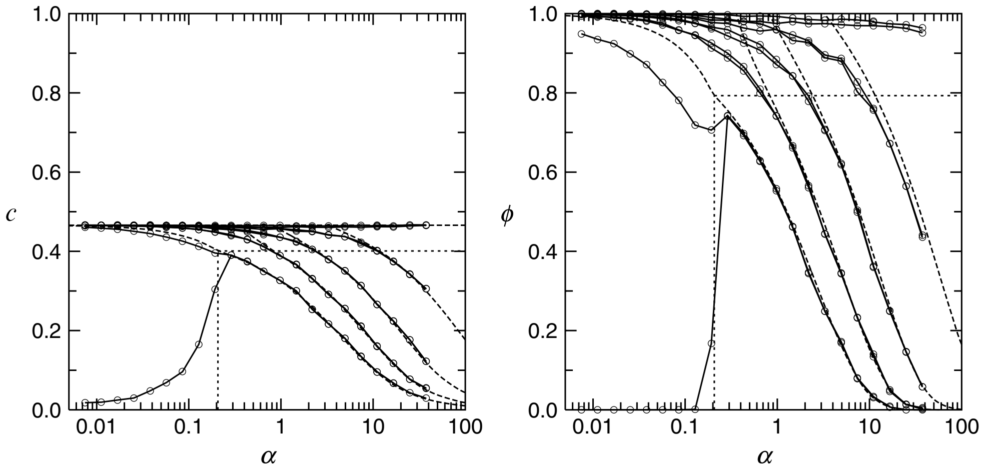


Fig. 7.5 The predicted correlations c (dashed curves, left picture) and the predicted fraction ϕ of frozen agents (dashed curves, right picture), together with simulation data (connected markers), for the batch MG with multiplicative decision noise of strength $T = 1$ and with self-impact correction. Self-impact correction factors: $\rho \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ (lower to upper curves in the high α regime). Initial conditions: $|q_i(0)| = \Delta$ for all i, data are shown for $\Delta \in \{0, 10\}$. The vertical dotted line marks the critical value $\alpha_c \approx 0.206930$ for the $\chi \to \infty$ transition (for $\rho = 0$ only). The horizontal line segments mark the predicted occurrence of the MO instability (with deviations between TTI theory and data predicted above these segments, but not below: $c^\star \approx 0.401$ and $\phi^\star \approx 0.793$).

we were to have $|q_i(t)| \to \infty$ for all i), do we find the volatility not being reduced to zero in the limit $\alpha \to 0$. As seen earlier in the simulation data for c and ϕ, for $\rho > 0$ the initial conditions also have a negligible influence on the volatility.

It seems that we may well interpret our results on the present model by stating that the main mechanism which was responsible for the high volatilities in the standard MG was the failure of the agents to correct for their own impact on the market, leading to instabilities caused by a negative effective feedback on their own actions. This is a remarkable and unexpected outcome. We appreciate once more the strength of the MG: its mathematical simplicity allows us to understand and predict such subtle mechanisms to a large extent quantitatively.

7.2 The spherical batch MG

7.2.1 Definitions and interpretation

We now turn to a second example of a modified MG where a novel type of non-equilibrium phase transition is found to emerge. Again our point of departure is the standard batch MG with two strategies per agent and with random market information,

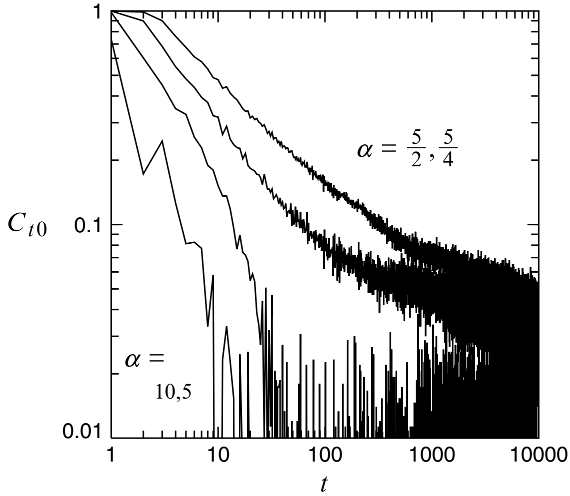


Fig. 7.6 Evolution of the correlation C_{t0} between the microscopic state at time t and the initial state, as measured in numerical simulations following initial conditions where $q_i(0) = 10$ for all i, and plotted logarithmically. Here $\rho = \frac{1}{2}$ (i.e. partial self-impact correction) and $T = 0$ (no decision noise), with $\alpha \in \{5, 10\}$ (in the TTI regime) and with $\alpha \in \{\frac{5}{4}, \frac{5}{2}\}$ (both in the non-TTI regime).

as defined by the microscopic equations (4.2) and (4.3) and with the decision noise definitions (2.12) and (2.13). In contrast to the previous model, which was at least partly motivated by real market considerations, here our motivation is purely mathematical: we seek a modification which will lead to an effective single-agent problem for which the order parameter equations for the kernels $\{C, G\}$ are exactly solvable, even for small values of α.

Experience with other disordered many-particle systems suggests that one way of achieving this objective is to construct a so-called spherical version, with microscopic equations for the N variables $q_i(t)$ which are *linear*, except for a global constraint $\sum_i q_i^2(t) = Nr^2$ which is to be satisfied at all times. In the batch MG this can be realized particularly easily: we first linearize the microscopic laws by replacing $\sigma[q_i(t), z_i(t)] \to q_i(t)$, and we subsequently subject the values $q_i(t+1)$ to an overall re-scaling such as to meet the requirement $\sum_i q_i^2(t+1) = Nr^2$. This gives us the following process

$$[1 + \lambda(t+1)]q_i(t+1) = q_i(t) + \theta_i(t) - \frac{2}{\sqrt{N}} \sum_{\mu=1}^{\alpha N} \xi_i^\mu A^\mu[\mathbf{q}(t)], \tag{7.47}$$

$$A^\mu[\mathbf{q}] = \Omega_\mu + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} q_i \xi_i^\mu, \quad r^2 = \frac{1}{N} \sum_i q_i^2(t). \tag{7.48}$$

The first line (7.47) represents the linearized microscopic equations, with a re-scaling parameter $\lambda(t)$ at each time step, which is to be solved from the second equation in

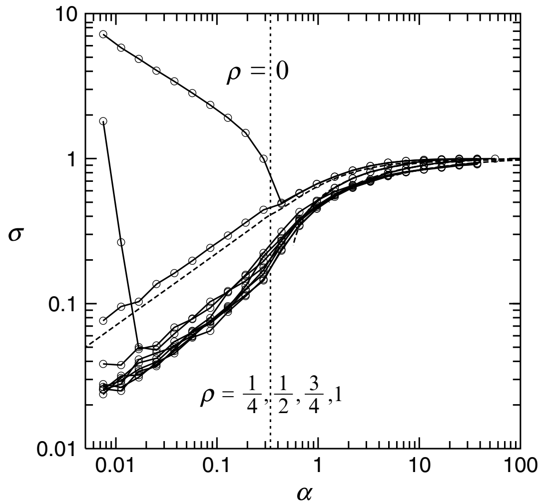


Fig. 7.7 The approximation (4.155) for the volatility σ in terms of persistent order parameters only (dashed curve), together with simulation data (connected markers), for the batch MG without decision noise but with self-impact correction. Self-impact correction factors: $\rho \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ (upper to lower curves in the high α regime). Initial conditions: $|q_i(0)| = \Delta$ for all i, data are shown for $\Delta \in \{0, 10\}$. The vertical dotted line marks the critical value $\alpha_c \approx 0.33740$ for the $\chi \to \infty$ transition (for $\rho = 0$ only).

(7.48).[37] The constraining forces induced by $\lambda(t)$ in equation (7.47) could in principle also generate overall sign changes, which we must exclude by insisting on $1+\lambda(t) > 0$ for all t. We see that we have lost[38] the decision noise parameter T, but we have gained a new control parameter r, which defines the radius of the sphere to which the microscopic state vector $\mathbf{q}$ is being confined.[39] A further consequence of the present definition is that the model (7.47) and (7.48) has no analogue of the *tabula rasa* MG initialization: the microscopic initial state where $q_i(0) = 0$ for all i is simply forbidden by the constraint $\sum_i q_i^2(0) = Nr^2$.

The spherical constraint is obviously a purely mathematical device, whose sole purpose is to provide the simplest possible mechanism for retaining non-linearity in our equations. The linearity of $A^\mu[\mathbf{q}]$ in equation (7.48), in contrast, turns out to have a natural and interesting interpretation. Let us recall the origin of the previous formulation with $\sigma[q_i, z_i]$ of our equations in the standard batch MG, which lay in the

[37] Without the spherical normalization the present model would be strictly linear, and therefore trivial, without any phase transitions.

[38] If required one could also quite easily define a solvable spherical model with additive decision noise, e.g. upon simply adding statistically independent zero-average Gaussian noise variables $\eta_i(t)$ to the external fields $\theta_i(t)$ at each time t, with $\langle \eta_i(t)\eta_j(t')\rangle = 2T\delta_{ij}\delta_{tt'}$. Here we will restrict ourselves to the simplest noise-free case.

[39] It could be argued that the 'natural' value to be chosen for r is one, given that in the ordinary batch MG we had $\sum_i \sigma^2[q_i(t), z_i(t)] = N$.

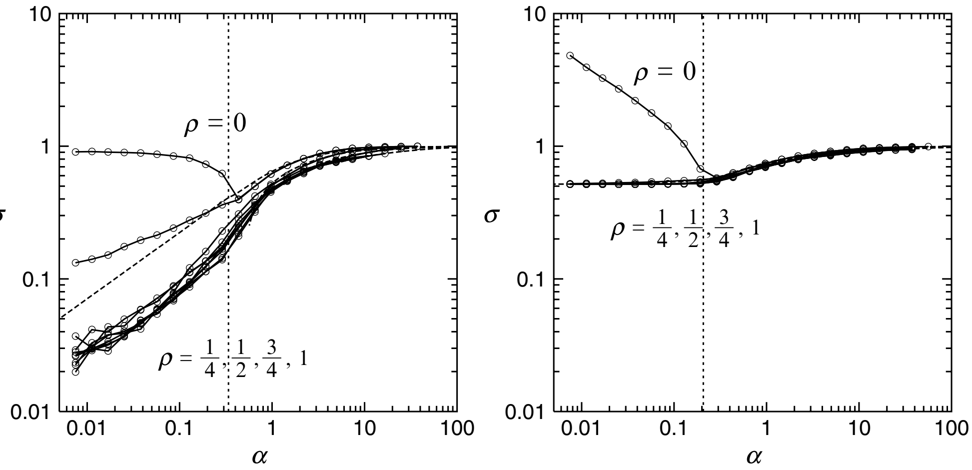


Fig. 7.8 The approximation (4.155) for the volatility σ in terms of persistent order parameters only (dashed curve), together with simulation data (connected markers), for the batch MG with decision noise of strength $T = 1$ (left figure: additive; right figure: multiplicative) and with self-impact correction. Self-impact correction factors: $\rho \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ (upper to lower curves in the high α regime). Initial conditions: $|q_i(0)| = \Delta$ for all i, data are shown for $\Delta \in \{0, 10\}$. The vertical dotted lines mark the critical values (left: $\alpha_c \approx 0.33740$; right: $\alpha_c \approx 0.206930$) for the $\chi \to \infty$ transition ($\rho = 0$ only).

expressions for the bids $b_i(t)$ of individual agents for $S = 2$:

$$\text{standard}: \quad b_i(t) = \frac{1}{2}\Big[1 + \sigma[q_i(t), z_i(t)]\Big] R^{i1}_{\mu(t)} + \frac{1}{2}\Big[1 - \sigma[q_i(t), z_i(t)]\Big] R^{i2}_{\mu(t)}$$

$$\text{spherical}: \quad b_i(t) = \frac{1}{2}[1 + q_i(t)]\, R^{i1}_{\mu(t)} + \frac{1}{2}[1 - q_i(t)]\, R^{i2}_{\mu(t)}.$$

Here $\mu(t)$ denotes the (random) fake history variable drawn at time t. We see that, whereas in all previous MG versions the agents could only act according to the prescriptions laid down in their given strategies (their only freedom being the selection of a strategy), the agents in the spherical batch MG will use *linear combinations* of their strategies, giving more weight to the best one rather than selecting the best one. They can even play the opposite of a strategy.

7.2.2 The effective single-agent process

As with the previous model with self-impact correction, the spherical MG is sufficiently similar to the ordinary batch MG to allow us to obtain the canonical effective single-agent equations without redoing the full generating functional analysis. We only have to be careful not to overlook any previous use of the statement $\sigma^2[q, z] = 1$, which is no longer correct if we replace $\sigma[q, z] \to q$. We will set up our calculation at first for any arbitrary given time dependence $\lambda(t)$, a unique trajectory will follow upon demanding self-consistency at the end (i.e. given the further demands $1+\lambda(t)>0$, only one choice will be compatible with the constraint $N^{-1}\sum_i \overline{q_i^2}=r^2$). As a first stage we see that the following three changes will occur in expression (4.16) for the generating functional: we find $s_i(t)$ replaced by $q_i(t)$ for all i and t, we find $i\hat{q}_i(t)q_i(t+1)$ replaced by $i[1+\lambda(t+1)]\hat{q}_i(t)q_i(t+1)$ for all i and t, and immediately after $\prod_{it}[\mathrm{d}q_i(t)\mathrm{d}\hat{q}_i(t)/2\pi]$ the following factor will now appear, which results from incorporating the spherical constraint by insertion of $\delta[Nr^2-\sum_i q_i^2(t)]$ for each time step t, and subsequently writing the various δ-functions in integral representation

$$\prod_t \frac{\mathrm{d}\hat{\lambda}(t)}{2\pi}\,\mathrm{e}^{\mathrm{i}\hat{\lambda}(t)[Nr^2-\sum_i q_i^2(t)]}. \tag{7.49}$$

Continuation of the batch calculation with these three modification leads as always to an expression for the disorder-averaged generating functional which can be evaluated by steepest descent. Here the relevant order parameters are:[40]

$$C_{tt'}=\lim_{N\to\infty}\frac{1}{N}\sum_i \overline{q_i(t)q_i(t')},\qquad G_{tt'}=\lim_{N\to\infty}\frac{1}{N}\sum_i \frac{\partial}{\partial\theta_i(t')}\overline{q_i(t)}, \tag{7.50}$$

in addition to the previous auxiliary macroscopic quantities $K_{tt'}$ and $L_{tt'}$ and the various conjugate kernels, but we also have the new quantities $\hat{\lambda}(t)$ which represent the spherical constraint. The disorder average of the batch MG is not affected by our modifications, and we find equation (4.23) being replaced by

$$\overline{Z[\boldsymbol{\psi}]}=\int[\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\prod_t \mathrm{d}\hat{\lambda}(t)\,\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(\log(N))}. \tag{7.51}$$

The function Φ in equation (7.51) is identical to its batch MG predecessor (4.25), but the functions Ψ and Ω are found to be modified

$$\Psi=i\sum_{tt'}[\hat{C}_{tt'}C_{tt'}+\hat{K}_{tt'}K_{tt'}+\hat{L}_{tt'}L_{tt'}]+ir^2\sum_t\hat{\lambda}(t), \tag{7.52}$$

$$\begin{aligned}\Omega=\frac{1}{N}\sum_i\log\int&\mathcal{D}q\mathcal{D}\hat{q}\;p_0(q(0))\\ &\times\mathrm{e}^{\mathrm{i}\sum_t\left\{\hat{q}(t)[[1+\lambda(t+1)]q(t+1)-q(t)-\theta_i(t)]+\psi_i(t)q(t)-\hat{\lambda}(t)q^2(t)\right\}}\\ &\times\mathrm{e}^{-\mathrm{i}\sum_{tt'}[\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')+\hat{C}_{tt'}q(t)q(t')+\hat{K}_{tt'}q(t)\hat{q}(t')]}.\end{aligned} \tag{7.53}$$

As always we may use the fields $\{\psi_i(t),\theta_i(t)\}$ to confirm the validity of equation (7.50), after which we may put $\psi_i(t)\to 0$ and $\theta_i(t)\to\theta(t)$. The generating functional

[40] In the absence of decision noise the present equations are deterministic, so the definitions of correlation and response functions no longer involve averages over the N-agent process.

(7.51) is evaluated by steepest descent, and evaluation of the saddle-points of $\Psi+\Phi+\Omega$ results in the familiar manner in closed equations for C and G. Here, however, we have additional saddle-point equations, namely those obtained by variation of $\hat{\lambda}(t)$, which give the expected demands $r^2 = C_{tt}$ for all t. The final result of our adaptations of the standard batch MG analysis is the following effective single-agent equation

$$[1+\lambda(t+1)]q(t+1) = q(t) + \theta(t) - \alpha \sum_{t' \leq t} (\mathbf{I}+G)^{-1}_{tt'} q(t') + \sqrt{\alpha}\,\eta(t). \quad (7.54)$$

Here $\eta(t)$ is the usual zero-average Gaussian noise, characterized by the usual covariances $\langle \eta(t)\eta(t')\rangle = \Sigma_{tt'}$, with $D_{tt'} = 1 + C_{tt'}$:

$$\Sigma = (\mathbf{I}+G)^{-1} D (\mathbf{I}+G^{\dagger})^{-1}. \quad (7.55)$$

The matrices C and G and the constraining forces $\lambda(t)$ are the dynamical order parameters of the present problem, to be determined self-consistently by solving

$$C_{tt'} = \langle q(t)q(t')\rangle_\star, \quad G_{tt'} = \frac{\partial}{\partial\theta(t')}\langle q(t)\rangle_\star, \quad C_{tt} = r^2, \quad (7.56)$$

where the brackets $\langle \ldots \rangle_\star$ refer to averaging over the stochastic process (7.54).

Since the modifications of the generating functional formalism for the batch MG, which were induced by the present spherical version, did not affect the function Φ in equation (7.51), one can easily convince oneself that for $N \to \infty$ the expression for the volatility matrix $\overline{\Xi}$ (4.5) as derived for the ordinary batch MG is still correct for the spherical version, i.e. $\overline{\Xi} = \frac{1}{2}\Sigma$, with Σ as given by equation (7.55).

Finally, let us note that the above way of dealing with the spherical normalization is the so-called 'soft constraint': we ultimately only demand that the relation $\lim_{N\to\infty} N^{-1}\sum_i \overline{q_i^2(t)} = r^2$ be satisfied, rather than the more strict $N^{-1}\sum_i q_i^2(t) = r^2$. This makes our calculation much easier, but implies that we allow for small deviations from the sphere for individual disorder realizations. For $N \to \infty$ the differences between 'hard' and 'soft' constraining in models such as ours should vanish, in view of the anticipated scaling $N^{-1}\sum_i q_i^2(t) = N^{-1}\sum_i \overline{q_i^2(t)} + \mathcal{O}(N^{-\frac{1}{2}})$.

7.2.3 Explicit macroscopic equations and solutions

We can now appreciate the mathematical advantages of the spherical formulation of the MG: since equation (7.54) is linear in the variables $\{q(t)\}$, here one can find the order parameter equation (7.56) in fully *explicit* form. To do so one just needs the simple identity $\langle \eta(t)q(t')\rangle_\star = \sqrt{\alpha}\sum_s \Sigma_{ts}G_{t's}$, which is derived easily via integration by parts over the Gaussian variables $\{\eta(t)\}$, and has been encountered and used before.

An explicit equation for C is obtained upon multiplying both sides of (7.54) by $q(t')$ and subsequently averaging over the disorder-generated noise. To deal with G, which as always obeys the causality condition $G_{tt'} = 0$ for $t \leq t'$, one takes a derivative of (7.54) with respect to the external field $\theta(t')$, followed by averaging over the noise. In the final result we take the limit $\theta(t) \to 0$ for all t (the fields are no longer needed), and we find

$$[1 + \lambda(t+1)]\, C_{t+1,t'} = C_{tt'} + \alpha[(\mathbb{I} + G)^{-1} D (\mathbb{I} + G^{\dagger})^{-1} G^{\dagger}]_{tt'} \\ -\alpha[(\mathbb{I} + G)^{-1} C]_{tt'}, \tag{7.57}$$

$$[1 + \lambda(t+1)]\, G_{t+1,t'} = G_{tt'} - \alpha[(\mathbb{I} + G)^{-1} G]_{tt'} + \delta_{tt'}. \tag{7.58}$$

These equations are to be solved subject to the constraints $C_{tt} = r^2$ and $1 + \lambda(t) > 0$, for all $t \geq 0$, and are fully closed and explicit in $\{C, G, \lambda\}$.

Continuing further in the footsteps of the ordinary batch MG, we can calculate again the dynamic order parameters for short times iteratively, starting from $C_{00} = r^2$ and $G_{00} = 0$ and relying heavily on the causality of the response function. Here this calculation is much easier, as it no longer involves averages over the effective Gaussian noise but has been reduced via equations (7.57) and (7.58) to a relatively simple algebraic exercise. For instance

$$t = t' = 0: \quad [1 + \lambda(1)]\, C_{10} = (1 - \alpha) r^2, \tag{7.59}$$

$$[1 + \lambda(1)]\, G_{10} = 1, \tag{7.60}$$

$$t = 0,\ t' = 1: \quad [1 + \lambda(1)]\, r^2 = (1 - \alpha) C_{10} + \alpha G_{10}[1 + r^2]. \tag{7.61}$$

These expressions can be combined to produce

$$\lambda(1) = -1 + \sqrt{1 + \alpha^2 + \alpha(r^{-2} - 1)}, \tag{7.62}$$

$$C_{10} = (1 - \alpha) r^2 / \sqrt{1 + \alpha^2 + \alpha(r^{-2} - 1)}, \tag{7.63}$$

$$G_{10} = 1 / \sqrt{1 + \alpha^2 + \alpha(r^{-2} - 1)}. \tag{7.64}$$

With these expressions we can proceed, in turn, to the short-time effective noise covariances (7.55), and thereby also to the volatility matrix elements

$$\Sigma_{00} = 1 + r^2, \tag{7.65}$$

$$\Sigma_{10} = 1 - \frac{1 + \alpha r^2}{\sqrt{1 + \alpha^2 + \alpha(r^{-2} - 1)}}, \tag{7.66}$$

$$\Sigma_{11} = 1 + r^2 - \frac{2}{\sqrt{1 + \alpha^2 + \alpha(r^{-2} - 1)}} + \frac{1 - r^2(1 - 2\alpha)}{1 + \alpha^2 + \alpha(r^{-2} - 1)}. \tag{7.67}$$

This iteration can be carried out by hand for an arbitrary number of time steps (although this becomes prohibitively complicated for larger times), or it could be continued numerically.

Unlike all previous MG versions, the above exercise reveals that the values for our order parameters $\{C, G, \lambda\}$ are completely *independent* of the initial conditions: the effective single-trader state probabilities at $t = 0$ are irrelevant, provided they comply with the requirement $\langle q^2(0)\rangle = C_{00} = r^2$. Not only does the spherical MG not allow for 'tabula rasa' initialization, the initial conditions even drop out of the macroscopic theory. A further consequence of this is that, if multiple stationary state solutions of our macroscopic laws exist for a given choice of the control parameters (α, r), then only one of these can ever occur; the alternative must be ruled out by incompatibility with the short-time solution.

We may also inspect the solution of our order parameter equations (7.57) and (7.58) in leading order for small α and finite times

$$[1 + \lambda(t+1)]\, C_{t+1,t'} = C_{tt'} + \mathcal{O}(\alpha), \tag{7.68}$$

$$[1 + \lambda(t+1)]\, G_{t+1,t'} = G_{tt'} + \delta_{tt'} + \mathcal{O}(\alpha). \tag{7.69}$$

Upon combining equation (7.68) for the choice $t' = t$ with that corresponding to $t' = t + 1$, we immediately find $\lambda(t) = \mathcal{O}(\alpha)$. This, in turn, gives

$$C_{tt'} = r^2 + \mathcal{O}(\alpha), \quad G_{tt'} = \begin{cases} 1 + \mathcal{O}(\alpha) & \text{for} \quad t > t' \\ \mathcal{O}(\alpha) & \text{for} \quad t \le t', \end{cases} \tag{7.70}$$

$$\Sigma_{tt'} = [1 + r^2 + \mathcal{O}(\alpha)]\Big[\sum_{t' \le t} (\mathbf{1} + G)^{-1}_{tt'}\Big]^2 = \mathcal{O}(\alpha^2). \tag{7.71}$$

We conclude that for small α the system will be in a completely frozen state, with a divergent integrated response $\chi = \sum_{s>0} G_{t+s,t}$ and vanishing volatility.

7.2.4 Stationary state solutions

Let us now try to find the stationary and TTI solutions of the order parameter equations (7.57) and (7.58). Here this is much easier than it was in previous models, due to the present explicit form of these equations as opposed to a definition via an effective single-agent process. Thus we make the ansatz

$$C_{tt'} = C(t - t'), \quad G_{tt'} = G(t - t'), \quad \lambda(t) = \lambda. \tag{7.72}$$

The operators C and G will now commute. Equations (7.57) and (7.58) clearly allow for TTI solutions, and take the simplified form

$$\begin{aligned}[1 + \lambda]\, C(t+1) - C(t) = {} & \alpha[(\mathbf{1} + G)^{-1} D (\mathbf{1} + G^\dagger)^{-1} G^\dagger](t) \\ & -\alpha[(\mathbf{1} + G)^{-1} C](t), \end{aligned} \tag{7.73}$$

$$[1+\lambda]\,G(t+1)-G(t)=\delta_{t0}-\alpha[(\mathbb{1}+G)^{-1}G](t), \tag{7.74}$$

with the constraints $C(0)=r^2$ and $\lambda>-1$. The obvious next step is to carry out Fourier transforms.[41] We use the following notation

$$C(t)=\int_{-\pi}^{\pi}\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}\omega t}\tilde{C}(\omega),\quad \tilde{C}(\omega)=\sum_{t=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega t}C(t) \tag{7.75}$$

These definitions allow us to convert equations (7.73) and (7.74) into fully diagonal form:

$$\tilde{\Delta}(\omega)\tilde{C}(\omega)=\frac{\alpha\tilde{D}(\omega)\tilde{G}(\omega)^*}{|1+\tilde{G}(\omega)|^2}-\frac{\alpha\tilde{C}(\omega)}{1+\tilde{G}(\omega)}, \tag{7.76}$$

$$\tilde{\Delta}(\omega)\tilde{G}(\omega)=1-\frac{\alpha\tilde{G}(\omega)}{1+\tilde{G}(\omega)}, \tag{7.77}$$

where $\tilde{\Delta}(\omega)=(1+\lambda)\mathrm{e}^{\mathrm{i}\omega}-1$, and where $\tilde{G}(\omega)^*$ denotes the complex conjugate of $\tilde{G}(\omega)$. The constraint $C(0)=r^2$ becomes

$$\int_{-\pi}^{\pi}\mathrm{d}\omega\,\tilde{C}(\omega)=2\pi r^2. \tag{7.78}$$

In our analysis below we will need both the ordinary susceptibility χ, as well as the long-term response $\hat{\chi}$ to persistent oscillatory perturbations

$$\chi=\sum_{t\geq 0}G(t)=\tilde{G}(0),\quad \hat{\chi}=\sum_{t\geq 0}G(t)(-1)^t=\tilde{G}(\pi). \tag{7.79}$$

Finally, we note that in the TTI stationary state the volatility matrix $\overline{\Xi}$ is also TTI, $\overline{\Xi}_{tt'}=\overline{\Xi}(t-t')$. This allows us to express the volatility $\sigma^2=\overline{\Xi}(0)=\frac{1}{2}[(\mathbb{1}+G)^{-1}D(\mathbb{1}+G^\dagger)^{-1}](0)$ as

$$\sigma^2=\frac{1}{2(1+\chi)^2}+\frac{1}{2}\int_{-\pi}^{\pi}\frac{\mathrm{d}\omega}{2\pi}\frac{\tilde{C}(\omega)}{|1+\tilde{G}(\omega)|^2}. \tag{7.80}$$

We will now first inspect stationary states with finite χ, i.e. those for which the response function decays sufficiently fast. Since $\tilde{D}(\omega)=\tilde{C}(\omega)+2\pi\delta(\omega)$, which is

[41] This could also have been done in the previous batch models, but there it would not have helped us much, due the implicit nature of the previous order parameter equations.

obtained with help of the sum representation (A.18) of the δ-distribution, we may rewrite equation (7.76) alternatively as

$$[\tilde{\Delta}(\omega)|1+\tilde{G}(\omega)|^2+\alpha]\tilde{C}(\omega)=2\pi\alpha\chi\delta(\omega). \tag{7.81}$$

We note that $\lambda=\tilde{\Delta}(0)$, so that putting $\omega=0$ in equation (7.77) allows us to express the stationary Lagrange parameter λ in terms of χ:

$$\lambda=\frac{1+\chi(1-\alpha)}{\chi(1+\chi)}. \tag{7.82}$$

It now follows from equation (7.81), together with the observation that $\tilde{\Delta}(\omega)$ is only real-valued for $\omega\in\{0,\pi\}$ (since $\lambda+1>0$) that the transformed correlation function must necessarily be of the form $\tilde{C}(\omega)=2\pi c_0\delta(\omega)+2\pi c_1\delta(\omega-\pi)$, or

$$C(t)=c_0+c_1(-1)^t. \tag{7.83}$$

Substitution of this form for $\tilde{C}(\omega)$ into the spherical constraint (7.78) shows that the two pre-factors c_0 and c_1 in equation (7.83) are coupled via equation $c_0+c_1=r^2$, so the properly normalized solution is

$$\tilde{C}(\omega)=2\pi c_0\delta(\omega)+2\pi[r^2-c_0]\delta(\omega-\pi). \tag{7.84}$$

Thus only the two Fourier modes $\omega=0$ and $\omega=\pi$ are found to play a role in the stationary state, the other modes just describe transients. Insertion of equation (7.84) into equation (7.81), and combination with equation (7.82) and with the two equations for χ and $\hat{\chi}$ which are obtained by choosing $\omega=0$ and $\omega=\pi$ in (7.77), respectively, finally gives us a closed set of four scalar equations from which to solve the persistent order parameters $\{\chi,\hat{\chi},c_0,\lambda\}$:

$$c_0\left[\alpha+\lambda(1+\chi)^2\right]=\alpha\chi, \tag{7.85}$$

$$(r^2-c_0)\left[\alpha-(\lambda+2)(1+\hat{\chi})^2\right]=0, \tag{7.86}$$

$$\lambda=\frac{1+(1-\alpha)\chi}{\chi(1+\chi)},\quad \lambda+2=-\frac{1+(1-\alpha)\hat{\chi}}{\hat{\chi}(1+\hat{\chi})}. \tag{7.87}$$

All the phenomenology of the stationary state solutions with finite response is contained in the solution(s) of these coupled equations. Given a solution, insertion of equation (7.84) into equation (7.80) subsequently gives the following (exact) expression for the volatility, in terms of the persistent order parameters only

$$\sigma^2=\frac{1+c_0}{2(1+\chi)^2}+\frac{r^2-c_0}{2(1+\hat{\chi})^2}. \tag{7.88}$$

We must conclude from equation (7.86) that there are two distinct types of stationary states with finite χ: a frozen state, where $c_0=r^2$ (so that $C(t)=2\pi c_0$ for all t), and an oscillating state, where $c_0 < r^2$. In addition we have to allow for the possibility of states with anomalous response, i.e. with $\chi = \infty$. This gives us three classes of stationary state solutions, which we will now work out separately:

- *Oscillating states* (O):
  Here we have $c_0 < r^2$, and the equations from which to solve $\{\chi, \hat{\chi}, c_0, \lambda\}$ reduce to the following

$$c_0\left[\alpha + \lambda(1+\chi)^2\right] = \alpha\chi, \tag{7.89}$$

$$\alpha = (\lambda+2)(1+\hat{\chi})^2, \tag{7.90}$$

$$\lambda = \frac{1+(1-\alpha)\chi}{\chi(1+\chi)}, \quad \lambda + 2 = -\frac{1+(1-\alpha)\hat{\chi}}{\hat{\chi}(1+\hat{\chi})}. \tag{7.91}$$

This set allows for different types of solutions, one of which gives a negative susceptibility, which we reject on physical grounds. This leaves, firstly, the following solution, which satisfies $\lambda + 1 > 0$ for all values of α and gives a finite positive susceptibility for sufficiently large α:

$$\lambda = \alpha - 1 + 2\sqrt{\alpha}, \tag{7.92}$$

$$\chi = \frac{1-\alpha-\sqrt{\alpha}+\sqrt{2\alpha^{3/2}+\alpha^2}}{2\sqrt{\alpha}+\alpha-1}, \tag{7.93}$$

$$\hat{\chi} = -1/(1+\sqrt{\alpha}). \tag{7.94}$$

We note that $\{\lambda, \chi, \hat{\chi}\}$ and hence also c_0 are independent of r. The remaining solution has $\lambda = \alpha - 1 - 2\sqrt{\alpha}$, and therefore meets our requirement $\lambda + 1 > 0$ only for $\alpha > 4$. It turns out that such solutions are never realized. For $\alpha \to \infty$ one finds

$$\chi = \frac{1}{2\alpha} + \mathcal{O}(\alpha^{-3/2}), \quad \hat{\chi} = -1 + \frac{1}{\sqrt{\alpha}} + \mathcal{O}(\alpha^{-1}), \quad \lambda = \alpha + \sqrt{\alpha}$$

$$c_0 = \frac{1}{2\alpha^2} + \mathcal{O}(\alpha^{-3/2}), \quad \sigma = \frac{1}{2}\alpha r^2 + \mathcal{O}(\sqrt{\alpha}).$$

Both the oscillations in the correlations and the volatility are seen to increase with increasing α. The present solution breaks down either when $\chi \to \infty$, or when $c_0 \to r^2$. The corresponding mathematical conditions are found to be $\alpha = \alpha_{c,1}$, with $\chi < \infty$ for $\alpha > \alpha_{c,1}$, and $\alpha = \alpha_{c,2}(r)$, with $c_1 > 0$ for $\alpha > \alpha_{c,2}(r)$, respectively, where[42]

[42] The derivation of equation (7.95) is trivial, but to find equation (7.96) one has to make use of a specific factorization, in order to avoid having to solve high order polynomial equations. The fastest route is to first express c_0 in terms of α only, by substitution of equations (7.92) and (7.93) into equation (7.85). This leads to a rather messy fraction, but careful inspection shows that numerator and denominator of this fraction have the common factor $1 + [\alpha + \sqrt{2\alpha}]^{\frac{1}{2}}$. Elimination of this factor leaves the simple statement $c_0 = \frac{1}{2}[(1+\sqrt{\alpha})/[\alpha + 2\sqrt{\alpha}]^{\frac{1}{2}} - 1]$, from which equation (7.96) follows immediately.

$$\alpha_{\rm c,1} = 3 - 2\sqrt{2} \approx 0.1716, \tag{7.95}$$

$$\alpha_{\rm c,2}(r) = \left[1 - \frac{1 + 1/2r^2}{\sqrt{1 + 1/r^2}}\right]^2. \tag{7.96}$$

It follows that which type of breakdown will actually occur as α is lowered must depend on the control parameter r. We note that $\alpha_{\rm c,1} = \alpha_{\rm c,2}(r)$ at the cross-over value $r = r_{\rm c} = \sqrt{\alpha_{\rm c,1}/(1 - \alpha_{\rm c,1})} \approx 0.455$, and that $\alpha_{\rm c,2}(r) > \alpha_{\rm c,1}$ for $r < r_{\rm c}$ and $\alpha_{\rm c,2}(r) < \alpha_{\rm c,1}$ for $r > r_{\rm c}$. Hence we expect that, as α is lowered for any $r > r_{\rm c}$, the amplitude of the oscillations of the correlations remains positive until the critical value $\alpha_{\rm c,1}$ is reached and a transition occurs to a state with anomalous response. At that point one has

$$\lim_{\alpha \downarrow \alpha_{\rm c,1}} \sigma^2 = \frac{1}{3 - 2\sqrt{2}}\left[r^2 - \frac{3 - 2\sqrt{2}}{2(\sqrt{2} - 1)}\right]. \tag{7.97}$$

Conversely, for $r < r_{\rm c}$ the oscillation amplitude of the correlations vanishes at $\alpha_{\rm c,2}(r)$, before anomalous response sets in, and at that point we must expect the system to enter a frozen state with finite integrated response.

- *Frozen states* (F):
  Here $c_0 = r^2$, and we now find closed equations for the pair $\{\chi, \lambda\}$ only

$$\lambda\chi^2 + (2\lambda - \alpha/r^2)\chi + \lambda + \alpha = 0, \tag{7.98}$$

$$\lambda\chi^2 + (\alpha - 1 + \lambda)\chi - 1 = 0. \tag{7.99}$$

The oscillation susceptibility $\hat{\chi}$ is no longer required in the saddle-point equations, and is also seen to drop out of the volatility (7.88), as a consequence of $c_0 = r^2$. After straightforward but tedious bookkeeping, we can determine the solution of equations (7.98) and (7.99):

$$\lambda = -1 - \alpha + \frac{(2 + 1/r^2)\sqrt{\alpha}}{\sqrt{1 + 1/r^2}}, \tag{7.100}$$

$$\chi = [\sqrt{\alpha}\sqrt{1 + 1/r^2} - 1]^{-1}. \tag{7.101}$$

Upon inserting equation (7.101) and the relation $c_0 = r^2$ into equation (7.88), one finds that in the frozen state the volatility is given by

$$\sigma^2 = \frac{1}{2}\left[\sqrt{r^2 + 1} - r/\sqrt{\alpha}\right]^2. \tag{7.102}$$

Exactly at the point where the volatility becomes zero, upon reducing α, we find that also $\lambda = 0$ and that the susceptibility diverges, marking a transition to a frozen state with anomalous response. All this happens when $\alpha = \alpha_{\rm c,3}$, where

$$\alpha_{\rm c,3}(r) = r^2/(r^2 + 1). \tag{7.103}$$

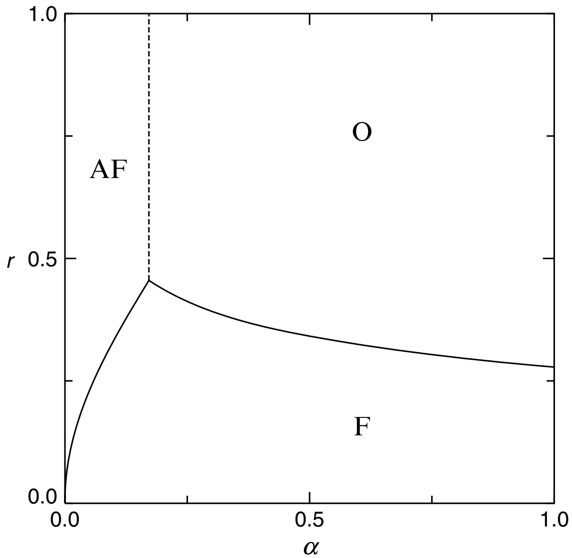


Fig. 7.9 Phase diagram in the (α, r) plane for the spherical MG with $S = 2$ and $\sum_i q_i(t)^2 = Nr^2$, exhibiting a frozen phase (F), an oscillating phase (O), and an anomalous frozen phase (AF) (see text for details). Solid lines indicate continuous transitions. The dashed line marks a discontinous transition. The triple point corresponds to $\alpha_c = 3 - 2\sqrt{2}$ and $r_c = [(3 - 2\sqrt{2})/(2\sqrt{2} - 2)]^{\frac{1}{2}}$.

- *Anomalous frozen states* (AF):
  We have seen that precisely at the transition line (7.103) one has $\chi = \infty$, $\lambda = 0$, and $\sigma = 0$. It turns out that a solution with these properties indeed solves our original dynamical equations self-consistently, also below the transition point (7.103), i.e. for $0 \leq \alpha < \alpha_{c,3}(r)$. We make the ansatz $C(t) = r^2$ for all t, together with $\lambda = 0$ and $\chi = \infty$ (which obviously satisfies our constraints). In Fourier language we now find

$$\tilde{C}(\omega) = 2\pi r^2 \delta(\omega), \quad \tilde{\Delta}(\omega) = e^{i\omega} - 1, \quad \tilde{D}(\omega) = 2\pi(1 + r^2)\delta(\omega).$$

Insertion into our original order parameter equations (7.76) and (7.77) gives

$$\delta(\omega)\, \frac{\tilde{G}(\omega)^* - r^2}{|1 + \tilde{G}(\omega)|^2} = 0, \tag{7.104}$$

$$(\mathrm{e}^{\mathrm{i}\omega} - 1)\tilde{G}(\omega) = 1 - \frac{\alpha \tilde{G}(\omega)}{1 + \tilde{G}(\omega)}. \tag{7.105}$$

For small ω the solution of the second equations is seen to be of the form $\tilde{G}(\omega) = (1 - \alpha)/i\omega + \mathcal{O}(\omega^0)$. Inserting this into the first equation subsequently gives an expression which is immediately seen to be true

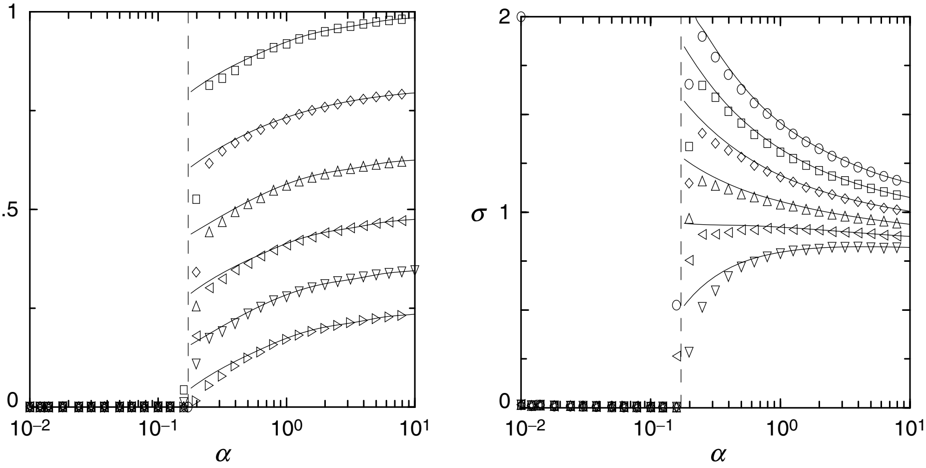


Fig. 7.10 Verification of the O→AF transition in the spherical MG. Left: the oscillation amplitude $c_1 = r^2 - c_0$ of the correlations versus α. Right: the volatility σ. In both cases $r \in \{0.5, 0.6, 0.7, 0.8, 0.9, 1\}$ (from bottom to top in both pictures). Markers: simulations; solid lines: theory. The dashed vertical lines mark the predicted location $\alpha_{c,1}$ of the (discontinuous) transition.

$$\delta(\omega)\,\frac{\mathrm{i}\omega + \mathcal{O}(\omega^2)}{1 - \alpha + \mathcal{O}(\omega)} = 0. \tag{7.106}$$

This confirms that the present ansatz, describing a fully frozen state with anomalous response, solves our order parameter equations (7.73) and (7.74).

7.2.5 The phase diagram and its verification

In summary, we have identified three distinct phases for the spherical MG in the (α, r) phase diagram, all three describing TTI stationary states, as well as the locations and nature (7.95), (7.96) and (7.103) of the phase transitions by which they are separated. Numerical simulations (see e.g. below) confirm that the system indeed always evolves into one of these phases

$$\begin{aligned}
\alpha > \max\{\alpha_{\mathrm{c},1},\ \alpha_{\mathrm{c},2}(r)\} :&\quad \text{oscillating phase (O)}\\
&\quad \chi < \infty,\ C(t) = c_0 + (r^2 - c_0)(-1)^t\\
&\quad \lambda \neq 0,\ \sigma > 0,\\
r < r_c,\ \alpha_{c,3}(r) < \alpha < \alpha_{c,2}(r) :&\quad \text{frozen phase (F)}\\
&\quad \chi < \infty,\ C(t) = r^2\\
&\quad \lambda \neq 0,\ \sigma > 0,
\end{aligned}$$

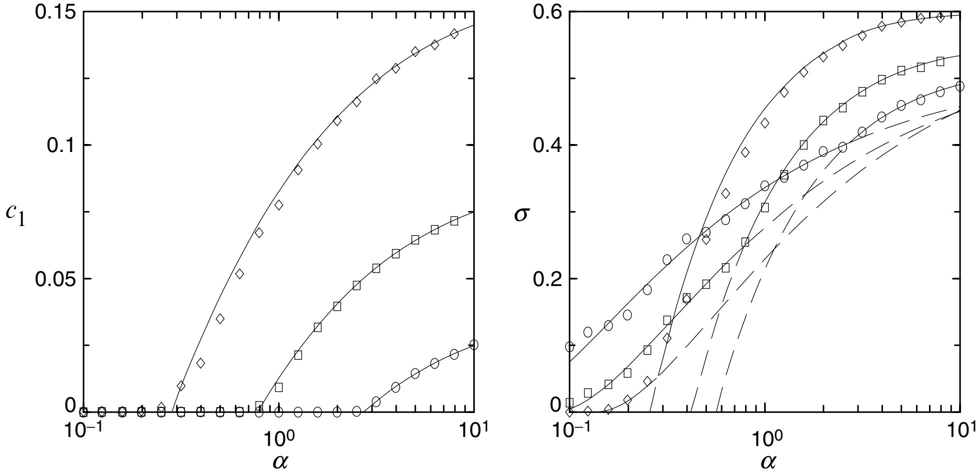


Fig. 7.11 Verification of the O→F transition in the spherical MG. Left: the oscillation amplitude $c_1 = r^2 - c_0$ of the correlations versus α. Right: the volatility σ. In both cases $r \in \{0.2, 0.3, 0.4\}$ (from bottom to top in both pictures). Markers: simulations; solid lines: theory. Here the transition point is r-dependent, and marks the destabilization of one solution in favour of another (the values of the order parameters of the phases O and F have been continued as dashed lines into the regimes where they are no longer stable).

$$\alpha < \min\{\alpha_{c,1},\ \alpha_{c,3}(r)\}: \quad \begin{array}{l} \text{anomalous frozen phase (AF)} \\ \chi = \infty,\ C(t) = r^2 \\ \lambda = 0,\ \sigma = 0. \end{array}$$

This information can be summarized in the phase diagram shown in Fig. 7.9. Due to the explicit nature of the order parameter equations (7.73) and (7.74) of the spherical MG (a crucial property, which was the whole point of the model's introduction in the first place), in each of the phases we have explicit expressions for all observables in the stationary state, including the volatility.

The location of the O→AF transition is given by equation (7.95). It follows already from equation (7.97), in combination with the property $\sigma = 0$ which we know holds in the AF phase, that this transition must be discontinuous. The system apparently 'collapses' into the fully frozen state at a specific critical value for α which is independent of the radius r of the constraining sphere. In contrast, the O→F transition given by equation (7.96) describes a continuous vanishing of the oscillation amplitude. Finally, the F→AF transition (7.103) is again continuous, marking the replacement of a frozen state (F) with fluctuations by one (AF) without any fluctuations. Numerical simulations of the spherical MG are found to give experimental results that are in perfect agreement with the above theoretical predictions, both in terms of the locations

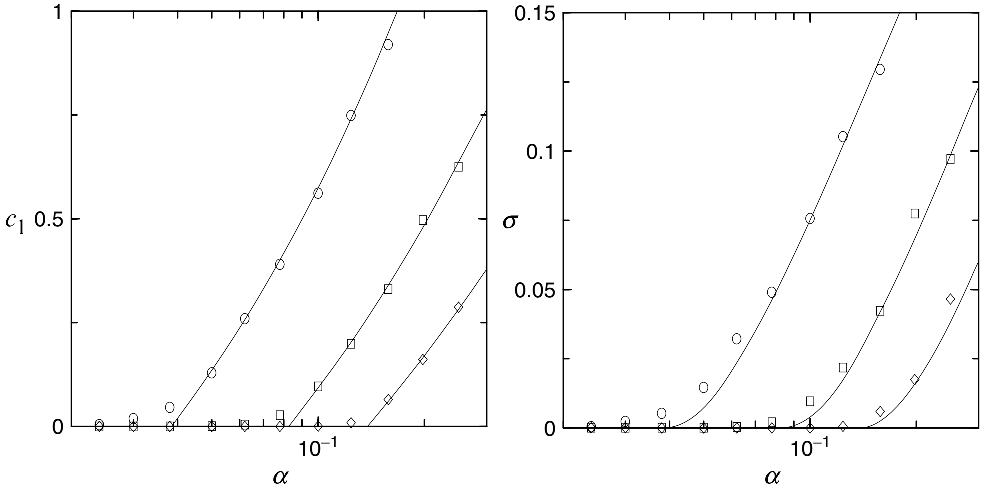


Fig. 7.12 Verification of the F→AF transition in the spherical MG. Left: the Lagrange parameter λ versus α. Right: the volatility σ versus α. In both cases $r \in \{0.2, 0.3, 0.4\}$ (from top to bottom in both pictures). Markers: simulations; solid lines: theory. The transition point is again r-dependent, and here marks the vanishing of both λ and σ (and the divergence of χ).

and nature of the predicted transitions and in terms of the actual values of the order parameters within the three phases. This can be inferred from Fig. 7.10 (focusing on the O→AF transition), Fig. 7.11 (focusing on the O→F transition), and Fig. 7.12 (focusing on the F→AF transition).

It will be clear that the phenomenology of the spherical MG is qualitatively different from that of the ordinary batch or on-line MGs, in several ways. Firstly, there is no dependence at all of the macroscopic observables on the initial conditions (provided the latter comply with the spherical constraint). Secondly, the volatility vanishes for $\alpha \to 0$, but diverges as $\alpha \to \infty$ (whereas in the ordinary MGs we always had $\lim_{\alpha\to\infty} \sigma = 1$). Thirdly, for sufficiently large values of the size r of the constraining sphere, one finds a *discontinuous* transition in the low α regime from an oscillating state (with non-zero volatility) to a rigorously frozen state, where $\sigma = 0$. As a technical exercise, the study of the spherical MG has certainly been fruitful. Whether and how such results can be used also in the context of the more standard MG versions is as yet unclear, but it would seem that at very least the effects of allowing agents to play linear combinations of their strategies would certainly be worth investigating further.

8. Dynamics of MGs with true market history

Except for the early chapters of this book, where we just presented numerical simulations in order to appreciate the phenomenology of the standard versions of MG, we have in all theoretical chapters so far carefully steered away from having to analyze MG models with true market history, and instead analyzed exclusively the mathematically more manageable Markovian 'fake history' versions. This was at least partly justified by the earlier observation in numerical simulations that the differences in behaviour between MGs with true market history and those with 'fake' history' were surprisingly small.

In this chapter we will finally return to the original on-line MG definition, still with $S = 2$ (two strategies per agent), but now with true external information which codes for the past history of the sign of the overall bid in the market. From a mathematical point of view the relevance of this exercise is clear: one would like to see whether and how the generating functional formalism (which has been successful for Markovian MGs) can be adapted to deal with history in the microscopic stochastic process, and also which type of effective single-agent equation (if any) will describe the MG with true history at the level of macroscopic order parameters, in the limit $N \to \infty$. From the more pragmatic point of view of those whose prime motivation is to understand the phenomenology of MGs there would also seem to be merit in trying to develop the theory for the case of true history: one would like to understand exactly why the differences between true and fake history in these models are as small as simulations suggest they are, since it is in no way a priori obvious that this should be so.

Since the batch versions of the MG were based on averaging by hand over all possible history strings at each stage of the process, before agents update their strategy valuations, it follows that batch dynamics is ruled out by definition if we wish to incorporate real market histories.

As in the previous applications of generating functional methods, the present analysis again consists of two distinct stages: we first work our way towards exact closed equations for dynamic order parameters, after which we concentrate on finding solutions for these closed laws. The main difference compared to with MGs with 'fake' histories will turn out to be that our closed dynamical theory is no longer defined in terms of one effective non-Markovian single-agent equation, but it now involves *two* such equations: one describing the dynamics of an effective agent and a second equation describing the effective dynamics of the overall bid.

8.1 Definitions

8.1.1 Restoration of market history

Thus we return to the appropriate stage in our general definitions immediately before we replaced the true market history by random numbers in the standard (on-line) MG, i.e. to the set of non-Markovian microscopic equations (2.2), (2.19)–(2.21), which also include the external contribution $A_{\rm e}(\ell)$ to the global bid (which here will play a useful role in our analysis) and now allowing for $\zeta < 1$:

$$q_i(\ell+1) = q_i(\ell) + \theta_i(\ell) - \frac{\tilde{\eta}}{N\sqrt{\alpha}} \sum_{\boldsymbol{\lambda}} \xi^i_{\boldsymbol{\lambda}} \mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] A(\ell), \tag{8.1}$$

$$A(\ell) = A_{\rm e}(\ell) + \frac{1}{\sqrt{\alpha N}} \sum_{\boldsymbol{\lambda}} \Big\{ \Omega_{\boldsymbol{\lambda}} + \frac{1}{\sqrt{N}} \sum_i \sigma[q_i(\ell), z_i(\ell)] \xi^i_{\boldsymbol{\lambda}} \Big\} \mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z], \tag{8.2}$$

$$\mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] = \sqrt{\alpha N}\, \delta_{\boldsymbol{\lambda}, \boldsymbol{\lambda}(\ell, A, Z)}, \tag{8.3}$$

$$\boldsymbol{\lambda}(\ell, A, Z) = \begin{pmatrix} \text{sgn}[(1-\zeta)A(\ell-1) + \zeta Z(\ell, 1)] \\ \vdots \\ \text{sgn}[(1-\zeta)A(\ell-M) + \zeta Z(\ell, M)] \end{pmatrix}. \tag{8.4}$$

The zero-average Gaussian random variables $\{z_i(\ell)\}$ (lower case Roman) represent decision noise, as before, with the choice made for the function $\sigma[q, z]$ controlling whether this noise is to act additively ($\sigma[q, z] = \text{sgn}[q + z]$) or multiplicatively ($\sigma[q, z] = \text{sgn}[q]\text{sgn}[1 + z]$). The parameter $\zeta \in [0, 1]$ allows us to interpolate between strictly true ($\zeta = 0$) and fully fake ($\zeta = 1$) market history; the relevant scaling regime for the history depth is $2^M = \alpha N$. All models studied so far have corresponded to $\zeta = 1$. The values of $\{A(\ell), Z(\ell)\}$ for $\ell \leq 0$ and of the $q_i(0)$ play the role of initial conditions.

The key differences at the mathematical level between MG models with fake history and those with true history as defined above, are in the dependence of the microscopic laws on the past via the history string $\{A(\ell-1), \ldots, A(\ell-M)\}$ occurring in $\boldsymbol{\lambda}(\ell, A, Z) \in \{-1, 1\}^M$, in combination with the presence and role of the zero-average Gaussian random variables $\{Z(\ell, \lambda)\}$ (upper case Roman). These latter variables generate effectively the fake (random) alternative[43] to the real market history $\{A(\ell-1),\ldots,A(\ell-M)\}$ in (8.4), in the sense of either (2.3) or (2.4).

In all cases of fake market history studied so far in this book, the choice $\zeta = 1$ eliminated with one stroke of the pen the dependence of the process on the history $\{A(\ell-1),\ldots,A(\ell-M)\}$, and the variables $\{Z(\ell,1),\ldots,Z(\ell,M)\}$ could and have subsequently been replaced simply by integer numbers μ, labelling each of the $2^M = p = \alpha N$ possible 'pseudo-histories' that could have been drawn at any given time step ℓ. Here all this is no longer possible. The variables $\{Z(\ell,\lambda)\}$ now play the role of random disturbances of the true market history as perceived by the agents in the game, and there is no reason why all 2^M possible histories should actually occur (let alone with equal frequencies) or why some entries $\{Z(\ell,\lambda)\}$ (e.g. those with small values of λ, which corrupt the most recent past in the history string) could not be more important than others. Our problem of solving the model mathematically has become *qualitatively* different, and it is therefore not a priori obvious that analytical progress can be made at all.

8.1.2 Further technicalities and definitions

One can anticipate various consequences for the generating functional analysis of introducing history into MG. An early appreciation of these will help us to proceed with the calculation more efficiently.

Firstly, since our objective is to analyze the original on-line version of the MG (rather than the batch version), we anticipate that the number of individual MG iterations required must be proportional[44] to N. However, the convenient temporal regularization method which we have employed successfully for the on-line MG with fake history, based on introducing Poissonnian distributed real-valued random durations for the individual iterations in equations (8.1) and (8.2), can in practice no longer be used in the non-Markovian case. The reason for this is the problem which prompted us earlier

[43] We have specifically defined our equations in such a manner that the main body of the calculation will also apply to alternative MG versions with history, such as those with the so-called inner-product definitions of the overall market bid, where strategies define linear functionals rather than look-up tables. The latter family of models would again require Gaussian variables representing the 'fake' market history, and would within the present notational conventions (and in its simplest form) just correspond to the replacements $\boldsymbol{\lambda} \to \lambda \in \{1,\ldots,\alpha N\}$ and $\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z] \to \mathcal{F}_{\lambda}[\ell,A,Z] = (1-\zeta)A(\ell-\lambda)+\zeta Z(\ell,\lambda)$.

[44] We will find in non-Markovian MGs that this need not always be true, but will rather depend on initial conditions, especially on the values assigned to the bids $A(\ell)$ with $\ell < 0$. The conventional scaling is correct when, for instance, the $A(\ell < 0)$ are drawn randomly and independently, with equal probabilities assigned to positive and negative values. If, on the other hand, we would choose all $A(\ell < 0)$ to be of the same sign, we would find the relevant processes to take place already on timescales where the number of iterations is of order $\sqrt{N}$. This can be interpreted as the MG eliminating structural bid predictibility (e.g. a situation where the overall bid is no longer zero on average) on much faster timescales.

to add the external perturbations $\theta_i(\ell)$ to the regularized on-line process rather than to the original equations, and replace equation (5.10) by equation (5.11): whereas in a strictly Markovian system the introduction of random durations for the individual iteration steps only implies a harmless uncertainty in where we are on our overall clock, in a system where we allow for retarded interactions we would generate very messy equations. We are therefore strongly encouraged to proceed with our process as it is, without temporal regularization (although we will be able to recover the previous theory in the limit $\zeta \to 1$, as it should),[45] and concentrate on the evaluation and disorder averaging of the generating functional

$$Z[\boldsymbol{\psi}] = \langle \mathrm{e}^{\mathrm{i}\sum_\ell \sum_i \psi_i(\ell)\sigma[q_i(\ell), z_i(\ell)]} \rangle. \tag{8.5}$$

The brackets in equation (8.5) denote averaging over the stochastic process (8.1) and (8.2), whose randomness is caused by the decision noise $\{\boldsymbol{z}(\ell)\}$ and the fake history variables $\{Z(\ell, \lambda)\}$. Although equation (8.5) looks like expression (4.9) for the batch MG, we note that here we have to allow for $\ell = \mathcal{O}(N)$.

Our choice to study the un-regularized process implies that initially we have to be more careful with finite size corrections. This has consequences in working out the disorder average of the generating functional: in the previous MG versions we needed only the first two moments of the distribution of the strategy look-up table entries $\{R^{ia}_{\boldsymbol{\lambda}}\}$. We chose to have $R^{ia}_{\boldsymbol{\lambda}} \in \{-1, 1\}$, with equal probabilities, but any alternative choice of distribution with the same average and variance would in leading order in N have given exactly the same equations. Here, although one should still expect only the first two moments to play a role in the final theory, the need to keep track initially of the finite size correction terms implies that our equations simplify considerably if, instead of binary strategy entries, we choose the variables $\{R^{ia}_{\boldsymbol{\lambda}}\}$ to be zero-average and unit-variance Gaussian variables.

Finally, our earlier definitions for the volatility matrix will no longer apply, as these were all based on being allowed to average over all possible 'pseudo-histories', at each time step (i.e. they all involved averages over all history labels μ). Here we have to return to basics and base our definitions on the actual values of the stochastically evolving bids $A(\ell)$ themselves, e.g.

$$\Xi_{\ell\ell'} = \langle A(\ell)A(\ell')\rangle - \langle A(\ell)\rangle\langle A(\ell')\rangle, \quad \sigma^2 = \lim_{L\to\infty} \frac{1}{L}\sum_{\ell=1}^{L} \Xi_{\ell\ell'}. \tag{8.6}$$

[45] It will in fact turn out that our present more direct application of the generating functional method, although perhaps somewhat more involved, does bring the benefit of greater transparency. For instance, the continuity assumptions underlying our use of saddle-point arguments in path integrals become much more transparent than they were in Chapter 5.

8.2 The disorder-averaged generating functional

It will turn out that in our analysis of equation (8.5) an important role will be played by the following quantity

$$\begin{aligned}\overline{W}[\ell,\ell';A,Z] &= \frac{1}{\alpha N}\sum_{\boldsymbol{\lambda}}\mathcal{F}[\ell,A,Z]\mathcal{F}[\ell',A,Z] \\ &= \delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)}. \end{aligned} \tag{8.7}$$

This binary object is a function of the paths $\{A\}$ and $\{Z\}$, and indicates whether or not the histories *as perceived by the agents* at times ℓ and ℓ' are identical (irrespective of the extent to which these 'histories' are true). Its statistics would have been trivial in the absence of history, but will now generally contain a significant amount of information regarding the recurrence of overall bid trajectories.

8.2.1 Evaluation of the disorder average

Rather than first writing the microscopic process in probabilistic form (which has been our standard procedure so far), we will here express the generating functional (8.5) as an integral over all possible joint paths of the state vector $\mathbf{q}$ and the overall bid A, and insert appropriate δ-distributions to enforce the microscopic dynamical equations[46] (8.1) and (8.2), i.e.

$$1 = \prod_{i\ell}\int\left[\frac{\mathrm{d}\hat{q}_i(\ell)}{2\pi}\right]\mathrm{e}^{\mathrm{i}\hat{q}_i(\ell)[q_i(\ell+1)-q_i(\ell)-\theta_i(\ell)+\frac{\tilde{\eta}}{N\sqrt{\alpha}}\sum_{\boldsymbol{\lambda}}\xi^i_{\boldsymbol{\lambda}}\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]]A(\ell)}$$

$$1 = \prod_{\ell}\int\left[\frac{\mathrm{d}\hat{A}(\ell)}{2\pi}\right]\mathrm{e}^{\mathrm{i}\hat{A}(\ell)[A(\ell)-A_{\mathrm{e}}(\ell)-\frac{1}{\sqrt{\alpha N}}\sum_{\boldsymbol{\lambda}}\left\{\Omega_{\boldsymbol{\lambda}}+\frac{1}{\sqrt{N}}\sum_i\sigma[q_i(\ell),z_i(\ell)]\xi^i_{\boldsymbol{\lambda}}\right\}\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]}.$$

To compactify our equations we will also use the shorthand which was introduced earlier, putting $s_i(\ell) = \sigma[q_i(\ell), z_i(\ell)]$. We can now write the disorder average $\overline{Z[\boldsymbol{\psi}]}$ of (8.5) as follows

$$\begin{aligned}\overline{Z[\boldsymbol{\psi}]} = \int&\left[\prod_{\ell>0}\frac{\mathrm{d}A(\ell)\mathrm{d}\hat{A}(\ell)}{2\pi}\mathrm{e}^{\mathrm{i}\hat{A}(\ell)[A(\ell)-A_{\mathrm{e}}(\ell)]}\right] \\ &\times\left\langle\int\left[\prod_{i\ell}\frac{\mathrm{d}q_i(\ell)\mathrm{d}\hat{q}_i(\ell)}{2\pi}\mathrm{e}^{\mathrm{i}\hat{q}_i(\ell)[q_i(\ell+1)-q_i(\ell)-\theta_i(\ell)]+i\psi_i(\ell)s_i(\ell)}\right]\right.\end{aligned}$$

[46] Since our microscopic laws are of an iterative and causal form, they have unique solutions and therefore the two procedures are fully equivalent.

$$\times \, \overline{\mathrm{e}^{\frac{\mathrm{i}}{N\sqrt{\alpha}}\sum_{\boldsymbol{\lambda}}\sum_{i\ell}\left[\tilde{\eta}\hat{q}_i(\ell)\xi^i_{\boldsymbol{\lambda}}A(\ell)-\hat{A}(\ell)\left(\omega^i_{\boldsymbol{\lambda}}+s_i(\ell)\xi^i_{\boldsymbol{\lambda}}\right)\right]\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]}} \Bigg\rangle_{\{\boldsymbol{z},Z\}} . \quad (8.8)$$

Here the brackets $\langle\ldots\rangle_{\{\mathbf{z},Z\}}$ denote averaging over the Gaussian decision noise and over the pseudo-memory variables, and we have also used the familiar abbreviations for combinations of look-up table entries

$$\omega^i_{\boldsymbol{\lambda}} = \frac{1}{2}(R^{i1}_{\boldsymbol{\lambda}} + R^{i2}_{\boldsymbol{\lambda}}), \quad \xi^i_{\boldsymbol{\lambda}} = \frac{1}{2}(R^{i1}_{\boldsymbol{\lambda}} - R^{i2}_{\boldsymbol{\lambda}}).$$

Using the shorthand $Du = (2\pi)^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}u^2}$ and the previously introduced quantity $\overline{W}[\ldots]$ in equation (8.7) allows us to write the disorder average (over the independently distributed zero-average and unit-variance $R^{ia}_{\boldsymbol{\lambda}}$) in the last line of (8.8) as

$$\begin{aligned}
\overline{\mathrm{e}^{\frac{\mathrm{i}}{N\sqrt{\alpha}}\sum_{\boldsymbol{\lambda}}\sum_{i\ell}\cdots}} &= \prod_{\boldsymbol{\lambda}}\prod_i \int Du\, \mathrm{e}^{\frac{\mathrm{i}u}{2N\sqrt{\alpha}}\sum_\ell\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)-\hat{A}(\ell)[1+s_i(\ell)]\right]\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]} \\
&\quad\times \prod_{\boldsymbol{\lambda}}\prod_i \int Dv\, \mathrm{e}^{\frac{\mathrm{i}v}{2N\sqrt{\alpha}}\sum_\ell\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)+\hat{A}(\ell)[1-s_i(\ell)]\right]\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]} \\
&= \mathrm{e}^{-\frac{1}{8N}\sum_i\sum_{\ell\ell'}\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)-\hat{A}(\ell)[1+s_i(\ell)]\right]\overline{W}[\ell,\ell';A,Z]\left[\tilde{\eta}\hat{q}_i(\ell')A(\ell')-\hat{A}(\ell')[1+s_i(\ell')]\right]} \\
&\quad\times \mathrm{e}^{-\frac{1}{8N}\sum_i\sum_{\ell\ell'}\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)+\hat{A}(\ell)[1-s_i(\ell)]\right]\overline{W}[\ell,\ell';A,Z]\left[\tilde{\eta}\hat{q}_i(\ell')A(\ell')+\hat{A}(\ell')[1-s_i(\ell')]\right]} \\
&= \mathrm{e}^{-\frac{1}{4N}\sum_{\ell\ell'>0}\overline{W}[\ell,\ell';A,Z]\sum_i\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)-\hat{A}(\ell)s_i(\ell)\right]\left[\tilde{\eta}\hat{q}_i(\ell')A(\ell')-\hat{A}(\ell')s_i(\ell')\right]} \\
&\quad\times \mathrm{e}^{-\frac{1}{4}\sum_{\ell\ell'>0}\hat{A}(\ell)\overline{W}[\ell,\ell';A,Z]\hat{A}(\ell')}.
\end{aligned} \quad (8.9)$$

We next isolate the usual observables $L(\ell,\ell') = N^{-1}\sum_i \hat{q}_i(\ell)\hat{q}_i(\ell')$, $K(\ell,\ell') = N^{-1}\sum_i s_i(\ell)\hat{q}_i(\ell')$, and $C(\ell,\ell') = N^{-1}\sum_i s_i(\ell)s_i(\ell')$, by inserting appropriate integrals over δ-distributions. We also use our familiar abbreviations $\mathcal{D}C = \prod_{\ell\ell'}[\sqrt{N}\mathrm{d}C(\ell,\ell')/\sqrt{2\pi}]$ (similarly for other two-time observables) and $\mathcal{D}A = \prod_{\ell>0}[\mathrm{d}A(\ell)/\sqrt{2\pi}]$ (similarly for $\hat{A}$). Initial conditions for the $q_i(0)$ are assumed to be of the factorized form $p_0(\mathrm{q}) = \prod_i p_0(q_i)$. In anticipation of issues to arise in subsequent stages of our analysis, especially those related to the scaling with N of the number of individual iterations of the process, we will also define the largest iteration step in the generating functional as $\ell_{\max}$. All this allows us to write $\overline{Z[\boldsymbol{\psi}]}$ in the form

$$\begin{aligned}
\overline{Z[\boldsymbol{\psi}]} &= \int \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}K\mathcal{D}\hat{K}\mathcal{D}L\mathcal{D}\hat{L}\; \mathrm{e}^{\mathrm{i}N\sum_{\ell\ell'}[\hat{C}(\ell,\ell')C(\ell,\ell')+\hat{K}(\ell,\ell')K(\ell,\ell')+\hat{L}(\ell,\ell')L(\ell,\ell')]} \\
&\quad\times \mathrm{e}^{\mathcal{O}(\ell_{\max}^2\log N)}\int \mathcal{D}A\mathcal{D}\hat{A}\; \mathrm{e}^{\mathrm{i}\sum_\ell \hat{A}(\ell)[A(\ell)-A_{\mathrm{e}}(\ell)]} \\
&\quad\times \mathrm{e}^{\frac{1}{4}\tilde{\eta}\sum_{\ell\ell'}\overline{W}[\ell,\ell';A,Z]\left\{\hat{A}(\ell)K(\ell,\ell')A(\ell')+\hat{A}(\ell')K(\ell',\ell)A(\ell)\right\}}
\end{aligned}$$

$$\times \ \mathrm{e}^{-\frac{1}{4}\sum_{\ell\ell'}\overline{W}[\ell,\ell';A,Z]\left\{\tilde{\eta}^2 A(\ell)L(\ell,\ell')A(\ell')+\hat{A}(\ell)[1+C(\ell,\ell')]\hat{A}(\ell')\right\}}$$
$$\times\left\langle\int\prod_{i\ell}\left[\frac{\mathrm{d}q_i(\ell)\mathrm{d}\hat{q}_i(\ell)}{2\pi}\mathrm{e}^{\mathrm{i}\hat{q}_i(\ell)[q_i(\ell+1)-q_i(\ell)-\theta_i(\ell)]+i\psi_i(\ell)s_i(\ell)}\right]\cdot\prod_i p_0(q_i(0))\right.$$
$$\left.\times\prod_i \mathrm{e}^{-\mathrm{i}\sum_{\ell\ell'}\{\hat{L}(\ell,\ell')\hat{q}_i(\ell)\hat{q}_i(\ell')+\hat{K}(\ell,\ell')s_i(\ell)\hat{q}_i(\ell')+\hat{C}(\ell,\ell')s_i(\ell)s_i(\ell')\}}\right\rangle_{\{\boldsymbol{z},Z\}}$$
$$=\int\mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}K\mathcal{D}\hat{K}\mathcal{D}L\mathcal{D}\hat{L}\ \mathrm{e}^{N[\Psi+\Omega+\Phi]+\mathcal{O}(\ell_{\max}^2\log N)} \tag{8.10}$$

with

$$\Psi = i\sum_{\ell\ell'\le\ell_{\max}}[\hat{C}(\ell,\ell')C(\ell,\ell')+\hat{K}(\ell,\ell')K(\ell,\ell')+\hat{L}(\ell,\ell')L(\ell,\ell')], \tag{8.11}$$

$$\Phi=\frac{1}{N}\log\left\langle\int\mathcal{D}A\mathcal{D}\hat{A}\ \mathrm{e}^{\mathrm{i}\sum_{\ell\le\ell_{\max}}\hat{A}(\ell)[A(\ell)-A_{\mathrm{e}}(\ell)]}\right.$$
$$\left.\times\ \mathrm{e}^{-\frac{1}{4}\sum_{\ell\ell'\le\ell_{\max}}\overline{W}[\ell,\ell';A,Z]M[\ell,\ell';A,\hat{A}]}\right\rangle_{\{Z\}}, \tag{8.12}$$

$$\Omega=\frac{1}{N}\sum_i\log\left\langle\int\left[\prod_{\ell=0}^{\ell_{\max}}\frac{\mathrm{d}q(\ell)\mathrm{d}\hat{q}(\ell)}{2\pi}\right]p_0(q(0))\right.$$
$$\times\ \mathrm{e}^{\mathrm{i}\sum_{\ell\le\ell_{\max}}\left[\hat{q}(\ell)[q(\ell+1)-q(\ell)-\theta_i(\ell)]+\psi_i(\ell)\sigma[q(\ell),z(\ell)]\right]-\mathrm{i}\sum_{\ell\ell'\le\ell_{\max}}\hat{q}(\ell)\hat{L}(\ell,\ell')\hat{q}(\ell')}$$
$$\left.\times\ \mathrm{e}^{-\mathrm{i}\sum_{\ell\ell'\le\ell_{\max}}\left[\hat{C}(\ell,\ell')\sigma[q(\ell),z(\ell)]\sigma[q(\ell'),z(\ell')]+\hat{K}(\ell,\ell')\sigma[q(\ell),z(\ell)]\hat{q}(\ell')\right]}\right\rangle_{\boldsymbol{z}}, \tag{8.13}$$

and with

$$M[\ell,\ell';A,\hat{A}]=\tilde{\eta}^2A(\ell)L(\ell,\ell')A(\ell')-\tilde{\eta}[\hat{A}(\ell)K(\ell,\ell')A(\ell')+\hat{A}(\ell')K(\ell',\ell)A(\ell)]$$
$$+\hat{A}(\ell)[1+C(\ell,\ell')]\hat{A}(\ell'). \tag{8.14}$$

We note that the $\mathcal{O}(\ell_{\max}^2\log N)$ corrections in (8.10) are only constants, which reflect the scaling with N used in defining the conjugate order parameters.

At this stage it is appropriate to reflect on the above results. Compared with our previous Markovian MG versions, we note that Ψ and Ω take their conventional forms, and that all the complications induced by having true market history are concentrated in the function $\Phi[C,K,L]$, which is now, interestingly, defined in terms of a stochastic process for the overall bid $A(\ell)$ rather than being an explicit function of the order parameters (which had been the situation in our previous fake history versions of the game), and in the remaining task to implement an appropriate scaling with N of the timescale $\ell_{\max}$.

We can now also appreciate the advantage offered by our earlier decision to define Gaussian rather than binary look-up table entries. With the N-scaling of $\ell_{\max}$ still pending, instead of (8.9), in the binary case we would have found

$$\overline{e^{\frac{i}{N\sqrt{\alpha}}\sum_{\boldsymbol{\lambda}}\sum_{i\ell}\cdots}} = e^{\sum_i\sum_{\boldsymbol{\lambda}}\log\cos\left[\frac{1}{2N\sqrt{\alpha}}\sum_{\ell\leq\ell_{\max}}\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)-\hat{A}(\ell)[1+s_i(\ell)]\right]\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]\right]} \times e^{\sum_i\sum_{\boldsymbol{\lambda}}\log\cos\left[\frac{1}{2N\sqrt{\alpha}}\sum_{\ell\leq\ell_{\max}}\left[\tilde{\eta}\hat{q}_i(\ell)A(\ell)+\hat{A}(\ell)[1-s_i(\ell)]\right]\mathcal{F}_{\boldsymbol{\lambda}}[\ell,A,Z]\right]}. \tag{8.15}$$

In this expression we see that, for $\ell_{\max} = \mathcal{O}(N)$, the different choices of strategy look-up table entry distribution will give the same results only for those paths $\{A, Z\}$ where the frequency of occurrence of each of the 2^M possible histories is of order $\mathcal{O}(N^{-1})$. In the latter case the function $\mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z]$ scales effectively inside summations over ℓ as $\mathcal{F}_{\boldsymbol{\lambda}}[\ell, A, Z] = \mathcal{O}(N^{-\frac{1}{2}})$, and we return to (8.9). Thus, for non-Gaussian distributions of the $\{R^{ia}_{\boldsymbol{\lambda}}\}$ at this stage of the generating functional analysis one either has to carry on with the more complicated expression (8.15), which cannot be expressed in terms of the order parameters $\{C, K, L\}$, or one has to make further assumptions on the overall bid statistics, which (although almost surely correct) will require careful validation a posteriori.

8.2.2 Canonical timescaling

We recall that in Chapter 2 we have already established that for the on-line MG with random external information (i.e. with $\zeta = 1$ in the present definitions) one must choose $\ell_{\max} = \mathcal{O}(N)$. Rather than just imposing this timescale $\ell_{\max} = \mathcal{O}(N)$ by hand, it is quite satisfactory to see that below we can also extract the canonical timescaling from our present equations (8.10)–(8.13).

For finite $\ell_{\max}$ we immediately find $\lim_{N\to\infty}\Phi = 0$ in (8.12), and our generating functional will be dominated by the physical saddle-point of $\lim_{N\to\infty}[\Psi + \Omega]$, giving $\hat{C} = \hat{K} = \hat{L} = 0$. This is seen to lead to a trivial effective single spin problem, which just describes a frozen state. This makes, of course, perfect sense in view of our definitions (8.1) and (8.2): since individual updates of the variables q_i are of order $N^{-\frac{1}{2}}$, nothing can change on timescales corresponding to only a finite number of iteration steps. Thus our present equations automatically lead us to the choice $\ell_{\max} = \mathcal{O}(1/\delta_N)$, where $\lim_{N\to\infty}\delta_N = 0$. We note that the function Φ will indeed scale differently as soon as $\ell_{\max}$ is allowed to diverge with N.

We thus define $\ell_{\max} = t_{\max}/\delta_N$, where $0 \leq t_{\max} < \infty$ (of order N^0) and with $\lim_{N\to\infty}\delta_N = 0$. In order to obtain well-defined limits at the end in (8.11), we see that we have to re-scale our conjugate order parameters according to $(\hat{C},\hat{K},\hat{L}) \to {\delta_N}^2(\hat{C},\hat{K},\hat{L})$. Furthermore, for the perturbation fields $\{\theta_i,\psi_i\}$ to retain statistical significance they also will have to be re-scaled in the familiar manner, according to $(\theta_i,\psi_i) \to {\delta_N}^{-1}(\tilde{\theta}_i,\tilde{\psi}_i)$ (similar to what effectively happened in Chapter 5). The integrations over order parameters and conjugate order parameters in (8.10) will now become path integrals for $N \to \infty$.[47]

It will be convenient to introduce the following effective measure

$$\langle g[\{q,\hat{q},z\}]\rangle_\star = \lim_{N\to\infty} \frac{1}{N}\sum_i \frac{\int\prod_{\ell=1}^{t_{\max}/\delta_N}[\mathrm{d}q(\ell)\mathrm{d}\hat{q}(\ell)]\langle M_i[\{q,\hat{q},z\}]g[\{q,\hat{q},z\}]\rangle_{\boldsymbol{z}}}{\int\prod_{\ell=1}^{t_{\max}/\delta_N}[\mathrm{d}q(\ell)\mathrm{d}\hat{q}(\ell)]\langle M_i[\{q,\hat{q},z\}]\rangle_{\boldsymbol{z}}}, \tag{8.16}$$

$$M_i[\{q,\hat{q},z\}] = p_0(q(0))\mathrm{e}^{\mathrm{i}\delta_N\sum_{\ell=1}^{t_{\max}/\delta_N}\hat{q}(\ell)\left[\frac{q(\ell+1)-q(\ell)}{\delta_N}-\tilde{\theta}_i(\ell)\right]+\mathrm{i}\delta_N\sum_\ell\tilde{\psi}_i(\ell)\sigma[q(\ell),z(\ell)]}$$
$$\times\mathrm{e}^{-\mathrm{i}{\delta_N}^2\sum_{\ell\ell'=1}^{t_{\max}/\delta_N}[\hat{L}(\ell,\ell')\hat{q}(\ell)\hat{q}(\ell')+\hat{K}(\ell,\ell')\sigma[q(\ell),z(\ell)]\hat{q}(\ell')+\hat{C}(\ell,\ell')\sigma[q(\ell),z(\ell)]\sigma[q(\ell'),z(\ell')]]}. \tag{8.17}$$

Now, upon substituting $\ell_{\max} = t_{\max}/\delta_N$ into our equations (8.11)–(8.13), followed by the appropriate re-scaling of the conjugate order parameters, these three functions acquire the following form (modulo irrelevant constants)

$$\Psi = i{\delta_N}^2\sum_{\ell\ell'\leq t_{\max}/\delta_N}\Big[\hat{C}(\ell,\ell')C(\ell,\ell')+\hat{K}(\ell,\ell')K(\ell,\ell')+\hat{L}(\ell,\ell')L(\ell,\ell')\Big], \tag{8.18}$$

$$\Phi = \frac{1}{N}\log\left\langle\int\mathcal{D}A\mathcal{D}\hat{A}\,\mathcal{W}[A,\hat{A}|Z]\right\rangle_{\{Z\}}, \tag{8.19}$$

$$\Omega = \frac{1}{N}\sum_i\log\int\prod_{\ell=1}^{t_{\max}/\delta_N}[\mathrm{d}q(\ell)\mathrm{d}\hat{q}(\ell)]\;\langle M_i[\{q,\hat{q},z\}]\rangle_{\boldsymbol{z}}. \tag{8.20}$$

with

$$\mathcal{W}[A,\hat{A}|Z] = \mathrm{e}^{\mathrm{i}\sum_{\ell=1}^{t_{\max}/\delta_N}\hat{A}(\ell)[A(\ell)-A_{\mathrm{e}}(\ell)]-\frac{1}{4}\sum_{\ell\ell'=1}^{t_{\max}/\delta_N}\overline{W}[\ell,\ell';A,Z]M[\ell,\ell';A,\hat{A}]}. \tag{8.21}$$

It is clear that Ψ and Ω now have proper $N \to \infty$ limits. Within our theory the canonical choice of δ_N is subsequently determined by the mathematical condition that

[47] This is the point, therefore, where the expected continuity assumptions regarding our macroscopic observables enter the scene. It is nice to see how in the present derivation these assumptions take a more transparent form than in Chapter 5, where they were somewhat hidden inside the details of the temporal regularization.

$\lim_{N\to\infty}\Phi[C,K,L]\neq 0$, but finite. It follows that (8.10) is now again dominated by its physical saddle-point, and we are nearly back in familiar territory.

8.2.3 The saddle-point equations

In order to eliminate the fields $\{\psi_i(\ell),\theta_i(\ell)\}$, and thereby simplify our subsequent equations, we next extract the physical meaning of our order parameters from the generating functional by taking appropriate derivatives with respect to these fields. This gives

$$C(\ell,\ell') = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle s_i(\ell)s_i(\ell')\rangle} = \langle\sigma[q(\ell),z(\ell)]\sigma[q(\ell'),z(\ell')]\rangle_\star, \quad (8.22)$$

$$G(\ell,\ell') = \lim_{N\to\infty}\frac{1}{N}\sum_i \frac{\partial\overline{\langle s_i(\ell)\rangle}}{\partial\theta_i(\ell')} = -i\langle\sigma[q(\ell),z(\ell)]\hat{q}(\ell')\rangle_\star, \quad (8.23)$$

$$0 = \lim_{N\to\infty}\frac{1}{N}\sum_i \frac{\partial^2 1}{\partial\theta_i(\ell)\partial\theta_i(\ell')} = -\langle\hat{q}(\ell)\hat{q}(\ell')\rangle_\star. \quad (8.24)$$

Thus at the physical saddle-point of (8.10) we must have the usual relations $L(\ell,\ell')=0$ and $K(\ell,\ell')=iG(\ell,\ell')$, where G denotes the disorder-averaged single-site response function. Upon varying $\{\hat{C},\hat{K},\hat{L}\}$ in (8.10) we see that we do indeed reproduce self-consistently the by now standard equations

$$C(\ell,\ell') = \langle\sigma[q(\ell),z(\ell)]\sigma[q(\ell'),z(\ell')]\rangle_\star, \quad (8.25)$$

$$G(\ell,\ell') = -i\langle\sigma[q(\ell),z(\ell)]\hat{q}(\ell')\rangle_\star, \quad (8.26)$$

$$L(\ell,\ell') = \langle\hat{q}(\ell)\hat{q}(\ell')\rangle_\star = 0. \quad (8.27)$$

We turn to variation of the order parameters $\{C,K,L\}$ in $\Psi+\Phi$ (as Ω only depends on the conjugate order parameters). In working out derivatives of Φ we observe that the conjugate bids effectively act as differential operators, i.e. $\hat{A}(s)\to i\partial/\partial A_{\rm e}(s)$. Here we can benefit from the earlier introduction of the external contributions $\{A_{\rm e}(s)\}$ to the overall bids. This gives us our remaining three saddle-point equations

$$\hat{C}(s,s') = \lim_{N\to\infty}\frac{i}{4N{\delta_N}^2}\,\frac{(\partial^2/\partial A_{\rm e}(s)\partial A_{\rm e}(s'))\left\langle\int\mathcal{D}A\mathcal{D}\hat{A}\,\mathcal{W}[A,\hat{A}|Z]\,\overline{W}[s,s';A,Z]\right\rangle_{\{Z\}}}{\left\langle\int\mathcal{D}A\mathcal{D}\hat{A}\,\mathcal{W}[A,\hat{A}|Z]\right\rangle_{\{Z\}}}, \quad (8.28)$$

$$\hat{K}(s,s') = \lim_{N\to\infty}\frac{-\tilde{\eta}}{2N{\delta_N}^2}\,\frac{(\partial/\partial A_{\rm e}(s))\left\langle\int\mathcal{D}A\mathcal{D}\hat{A}\,\mathcal{W}[A,\hat{A}|Z]\,\overline{W}[s,s';A,Z]A(s')\right\rangle_{\{Z\}}}{\left\langle\int\mathcal{D}A\mathcal{D}\hat{A}\,\mathcal{W}[A,\hat{A}|Z]\right\rangle_{\{Z\}}}, \quad (8.29)$$

$$\hat{L}(s,s') = \lim_{N\to\infty} \frac{-i\tilde{\eta}^2}{4N\delta_N{}^2} \frac{\left\langle \int \mathcal{D}A\mathcal{D}\hat{A}\, \mathcal{W}[A,\hat{A}|Z]\, \overline{W}[s,s';A,Z]A(s)A(s')\right\rangle_{\{Z\}}}{\left\langle \int \mathcal{D}A\mathcal{D}\hat{A}\, \mathcal{W}[A,\hat{A}|Z]\right\rangle_{\{Z\}}}. \qquad (8.30)$$

At the physical saddle-point, we may use $L = 0$ and the symmetry of $\overline{W}[\ldots]$ to simplify the function $M[\ell,\ell';A,\hat{A}]$ which occurs in the measure (8.21) to

$$M[\ell,\ell';A,\hat{A}] = \hat{A}(\ell)[1+C(\ell,\ell')]\hat{A}(\ell') - 2i\tilde{\eta}\hat{A}(\ell)G(\ell,\ell')A(\ell'). \qquad (8.31)$$

The generating fields $\{\tilde{\psi}_i(\ell)\}$ are now no longer needed and can be put to zero. The perturbations $\tilde{\theta}_i$ are still useful for calculating the response function G, but can now be chosen to be site-independent, i.e. $\tilde{\theta}_i(\ell) = \tilde{\theta}(\ell)$. The measure (8.17) will consequently lose its site-dependence. Also the three functions $\{\Psi, \Phi, \Omega\}$ have at this stage become obsolete. We may define a new time $t = \ell\delta_N = \mathcal{O}(N^0)$, which will be real-valued in the limit $N \to \infty$, and we may take the limit $N \to \infty$ in the definitions of our observables. The latter can subsequently be written in terms of the new real-valued time arguments simply as $C(\ell,\ell') \to C(t,t')$, and similar for the other kernels.

8.3 The resulting theory

8.3.1 Simplification of saddle-point equations

We may now summarize our saddle-point equations for $\{C, G\}$ in the usual compact way, in terms of an effective single-agent process

$$C(t,t') = \langle \sigma[q(t),z(t)]\sigma[q(t'),z(t')]\rangle_\star, \quad G(t,t') = -i\langle \sigma[q(t),z(t)]\hat{q}(t')\rangle_\star \qquad (8.32)$$

with a measure which is defined in terms of path integrals, as in Chapter 5 (and with time integrals running from $t = 0$ to $t = t_{\max}$):

$$\langle g[\{q,\hat{q},z\}]\rangle_\star = \frac{\int\{\mathrm{d}q\mathrm{d}\hat{q}\}\ \langle M[\{q,\hat{q},z\}]\ g[\{q,\hat{q},z\}]\rangle_{\mathbf{z}}}{\int\{\mathrm{d}q\mathrm{d}\hat{q}\}\ \langle M[\{q,\hat{q},z\}]\rangle_{\mathbf{z}}} \qquad (8.33)$$

$$M[\{q,\hat{q},z\}] = p_0(q(0))\ \mathrm{e}^{\mathrm{i}\int \mathrm{d}t\ \hat{q}(t)\left[\frac{\mathrm{d}}{\mathrm{d}t}q(t)-\theta(t)-\int \mathrm{d}t'\hat{K}(t',t)\sigma[q(t'),z(t')]\right]}$$
$$\times\ e^{-i\int \mathrm{d}t\mathrm{d}t'\left[\hat{L}(t,t')\hat{q}(t)\hat{q}(t')+\hat{C}(t,t')\sigma[q(t),z(t)]\sigma[q(t'),z(t')]\right]}. \qquad (8.34)$$

In order to find the kernels $\{\hat{C}, \hat{K}, \hat{L}\}$ we have to evaluate the saddle-point equations (8.28)–(8.30) further, remembering that the left-hand sides as yet still involve the integer time labels (s, s'), rather than the above continuous times. Now the scaling chosen for δ_N with N which we adopt will turn out to be crucial. We observe that all complications are contained in the evaluation, for large N and for any given realization of the fake market information path $\{Z\}$, of objects of the following general form (with all operators evaluated at the saddle-point)

$$\langle Q[\{A\}]\rangle_{\{A|Z\}} = \int \mathcal{D}A\mathcal{D}\hat{A}\ \mathcal{W}[A,\hat{A}|Z]\ Q[\{A\}]. \tag{8.35}$$

In fact we can quite easily confirm, by following the same steps taken so far in evaluating the disorder-averaged generating functional $\overline{Z[\psi]}$ but now for calculating averages of arbitrary functions of the overall market bid path $\{A\}$, that the physical interpretation of the measure (8.35) is

$$\lim_{N\to\infty} \overline{\langle Q[\{A\}]\rangle} = \left\langle \langle Q[\{A\}]\rangle_{\{A|Z\}} \right\rangle_{\{Z\}}. \tag{8.36}$$

Thus equation (8.35) defines the asymptotic disorder-averaged probability density for observing a 'path' $\{A\}$ of global market histories, for a given realization of the fake history path $\{Z\}$.

In order to evaluate (8.35) we first define the two path-dependent matrices $G[A,Z]$ and $D[A,Z]$, with entries

$$G[A,Z](\ell,\ell') = \overline{W}[\ell,\ell';A,Z]G(\ell,\ell'), \tag{8.37}$$

$$D[A,Z](\ell,\ell') = \overline{W}[\ell,\ell';A,Z][1+C(\ell,\ell')]. \tag{8.38}$$

The definition (8.7) of $\overline{W}[\ldots]$ tells us that $G[A,Z](\ell,\ell') = G(\ell,\ell')$ if the 'history' observed at stage ℓ is identical to that observed at stage ℓ', and zero otherwise, and similarly for the relation between the values of $D[A,Z](\ell,\ell')$ and those of $1+C(\ell,\ell')$. We now use auxiliary integration variables $\{\phi_\ell\}$ to linearize the term in the exponent of (8.35) which is quadratic in $\hat{A}$, and use causality of the response function G where appropriate

$$\begin{aligned}
\langle Q[\{A\}]\rangle_{\{A|Z\}} &= \int \prod_{\ell=1}^{t_{\max}/\delta_N}\left[\frac{\mathrm{d}A(\ell)\mathrm{d}\hat{A}(\ell)}{2\pi}\mathrm{e}^{\mathrm{i}\hat{A}(\ell)[A(\ell)-A_{\mathrm{e}}(\ell)+\frac{1}{2}\tilde{\eta}\sum_{\ell'<\ell}G[A,Z](\ell,\ell')A(\ell')]}\right]Q[\{A\}]\\
&\quad\times\frac{\int[\prod_{\ell=1}^{t_{\max}/\delta_N}\mathrm{d}\phi_\ell]\mathrm{e}^{-\sum_{\ell\ell'=1}^{t_{\max}/\delta_N}\phi_\ell(D^{-1}[A,Z])_{\ell\ell'}\phi_{\ell'}-i\sum_{\ell=1}^{t_{\max}/\delta_N}\phi_\ell\hat{A}_\ell}}{\int[\prod_\ell\mathrm{d}\phi_\ell]\mathrm{e}^{-\sum_{\ell\ell'=1}^{t_{\max}/\delta_N}\phi_\ell(D^{-1}[A,Z])_{\ell\ell'}\phi_{\ell'}}}\\
&= \int\left[\prod_{\ell=1}^{t_{\max}/\delta_N}\mathrm{d}A(\ell)\right]Q[\{A\}]\\
&\quad\times\left\langle\prod_{\ell=1}^{t_{\max}/\delta_N}\int\frac{\mathrm{d}\hat{A}}{2\pi}\mathrm{e}^{\mathrm{i}\hat{A}[A(\ell)-A_{\mathrm{e}}(\ell)+\frac{1}{2}\tilde{\eta}\sum_{\ell'<\ell}G[A,Z](\ell,\ell')A(\ell')-\phi_\ell]}\right\rangle_{\{\phi|A,Z\}}
\end{aligned}$$

$$= \int \left[\prod_{\ell=1}^{t_{\max}/\delta_N} \mathrm{d}A(\ell) \right] Q[\{A\}]$$
$$\times \left\langle \prod_{\ell=1}^{t_{\max}/\delta_N} \delta \left[A(\ell) - A_{\mathrm{e}}(\ell) + \frac{1}{2}\tilde{\eta} \sum_{\ell'<\ell} G[A,Z](\ell,\ell')A(\ell') - \phi_\ell \right] \right\rangle_{\{\phi|A,Z\}}$$

Here $\langle \ldots \rangle_{\{\phi|A,Z\}}$ refers to averaging over the zero-average Gaussian fields ϕ_ℓ with $\{A,Z\}$-dependent covariance $\langle \phi_\ell \phi_{\ell'} \rangle_{\{\phi|A,Z\}} = \frac{1}{2} D[A,Z](\ell,\ell')$. We conclude from the above expression for $\langle Q[\{A\}] \rangle_{\{A|Z\}}$ that the conditional disorder-averaged probability density $\mathcal{P}[\{A\}|\{Z\}]$ for finding a global bid path $\{A\}$, given a realization $\{Z\}$ of the pseudo-history, is given by

$$\mathcal{P}[\{A\}|\{Z\}]$$
$$= \left\langle \prod_{\ell=1}^{t_{\max}/\delta_N} \delta \left[A(\ell) - A_{\mathrm{e}}(\ell) + \frac{1}{2}\tilde{\eta} \sum_{\ell'<\ell} G(\ell,\ell')\overline{W}[\ell,\ell';A,Z]A(\ell') - \phi_\ell \right] \right\rangle_{\{\phi|A,Z\}} \tag{8.39}$$

and that hence $\langle Q[\{A\}] \rangle_{\{A|Z\}} = \int [\prod_\ell dA(\ell)] \mathcal{P}[\{A\}|\{Z\}] Q[\{A\}]$. Causality ensures that the probability density (8.39) is properly normalized, since both ϕ_ℓ and $G[A,Z](\ell,\ell')$ involve only entries of the paths $\{A,Z\}$ with times $k < \ell$.

Having established (8.39), our saddle-point equations (8.28)–(8.30) can be simplified considerably. We immediately find that $\hat{C} = 0$. To simplify comparison with the results in Chapter 5 (corresponding to $\zeta = 1$), we will make a final change in notation and carry out the following transformations

$$\hat{K}(\ell,\ell') = -\alpha R(\ell',\ell), \quad \hat{L}(\ell,\ell') = -\frac{1}{2}\alpha i \Sigma(\ell,\ell'). \tag{8.40}$$

This allows us, with $p = \alpha N$ and in anticipation of our expected timescaling $\delta_N = \tilde{\eta}/2p$ (known from our earlier analysis in Chapter 2 of the Markovian limit $\zeta = 1$), to write the remaining equations (8.29) and (8.30) in the simple form

$$R(\ell,\ell') = \lim_{N\to\infty} \frac{\partial}{\partial A_{\mathrm{e}}(\ell')} \left\{ \frac{\tilde{\eta}}{2p{\delta_N}^2} \left\langle \langle \overline{W}[\ell',\ell;A,Z]A(\ell) \rangle_{\{A|Z\}} \right\rangle_{\{Z\}} \right\}, \tag{8.41}$$

$$\Sigma(\ell,\ell') = \lim_{N\to\infty} \left\{ \frac{\tilde{\eta}^2}{2p{\delta_N}^2} \left\langle \langle \overline{W}[\ell,\ell';A,Z]A(\ell)A(\ell') \rangle_{\{A|Z\}} \right\rangle_{\{Z\}} \right\}. \tag{8.42}$$

We see that R defines a macroscopic response function associated with external bid perturbation via A_{e}, and hence obeys causality: $R(\ell,\ell') = 0$ for $\ell' > \ell$. This, in turn, also enables us to simplify our saddle-point equation (8.32) for $\{C,G\}$ and the measure $\langle\ldots\rangle_\star$ to a form identical to that found for the Markovian ('fake history') on-line MG in Chapter 5:

$$C(t,t') = \langle \sigma[q(t),z(t)]\sigma[q(t'),z(t')]\rangle_\star, \tag{8.43}$$

$$G(t,t') = \frac{\delta}{\delta\tilde{\theta}(t')}\langle \sigma[q(t),z(t)]\rangle_\star, \tag{8.44}$$

$$\langle g[\{q,z\}]\rangle_\star = \frac{\int\{\mathrm{d}q\}\ \langle g[\{q,z\}]\ M[\{q,z\}]\rangle_{\boldsymbol{z}}}{\int\{\mathrm{d}q\}\ \langle M[\{q,z\}]\rangle_{\boldsymbol{z}}}, \tag{8.45}$$

$$M[\{q,z\}] = p_0(q(0)) \int\{\mathrm{d}\hat{q}\}\ \mathrm{e}^{-\frac{1}{2}\alpha\int\mathrm{d}t\mathrm{d}t'\ \Sigma(t,t')\hat{q}(t)\hat{q}(t')} \times \mathrm{e}^{\mathrm{i}\int\mathrm{d}t\ \hat{q}(t)\left[\frac{\mathrm{d}}{\mathrm{d}t}q(t)-\tilde{\theta}(t)+\alpha\int\mathrm{d}t'\ R(t,t')\sigma[q(t'),z(t')]\right]}. \tag{8.46}$$

8.3.2 Summary and interpretation

We recognize that (8.46) describes the familiar type of effective single-trader equation with a retarded self-interaction and zero-average Gaussian noise $\eta(t)$ with non-trivial temporal correlations $\langle\eta(t)\eta(t')\rangle = \Sigma(t,t')$:

$$\frac{\mathrm{d}}{\mathrm{d}t}q(t) = \tilde{\theta}(t) - \alpha\int_0^t \mathrm{d}t'\ R(t,t')\ \sigma[q(t')] + \sqrt{\alpha}\ \eta(t). \tag{8.47}$$

In writing (8.47) we have used the fact, as was done also in Chapter 5, that the discontinuity of the correlation function for equal times, i.e. $C(t,t)=1$, will in the continuous time limit be irrelevant. This implies that we may carry out the averages over the decision noise and are left only with expressions involving $\sigma[q] = \int\mathrm{d}z\ P(z)\sigma[q,z]$, and that (with the exclusion of $t=t'$, where one has $C(t,t)=1$) the order parameter equations (8.43) and (8.44) simplify to

$$C(t,t') = \langle\sigma[q(t)]\sigma[q(t')]\rangle_\star, \quad G(t,t') = \frac{\delta}{\delta\tilde{\theta}(t')}\langle\sigma[q(t)]\rangle_\star. \tag{8.48}$$

Our remaining problem is to solve the order parameters $\{R,\Sigma\}$ from equations (8.41) and (8.42). To do so we must select the canonical timescale δ_N such that the $N\to\infty$ limit in (8.41) and (8.42) is both non-trivial (i.e. δ_N sufficiently small) and well-defined (i.e. δ_N not too small). For the special value $\zeta=1$ (giving the on-line MG without market history) we have found in Chapter 2 that $\delta_N = \tilde{\eta}/2p$. Although here we have followed a different route towards a continuous time description, we will re-confirm below that indeed $\delta_N = \tilde{\eta}/2p$, by working out our present equations in detail for the fake history limit $\zeta\to 1$. Given this canonical timescaling and given the definition

$\overline{W}[\ell,\ell';A,Z] = \delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)}$, we find our equations (8.41) and (8.42) taking their final forms

$$R(t,t') = \frac{\delta}{\delta A_{\rm e}(t')} \lim_{\delta_N\to 0} \left\langle\!\left\langle A(\ell)\delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)}\right\rangle\!\right\rangle_{\{A,Z\}}\Big|_{\ell=t/\delta_N,\ell'=t'/\delta_N}, \quad (8.49)$$

$$\Sigma(t,t') = \tilde{\eta} \lim_{\delta_N\to 0} \frac{1}{\delta_N} \left\langle\!\left\langle A(\ell)A(\ell')\delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)}\right\rangle\!\right\rangle_{\{A,Z\}}\Big|_{\ell=t/\delta_N,\ell'=t'/\delta_N} \quad (8.50)$$

(with $\delta/\delta A_{\rm e}(\ell) = {\delta_N}^{-1}\partial/\partial A(\ell)$). Here $\langle\langle\ldots\rangle\rangle_{A,Z}$ refers to an average over the stochastic process (8.39) for the overall bids $\{A\}$ and over the pseudo-history $\{Z\}$. The bid evolution process can be written in more explicit form as

$$A(\ell) = A_{\rm e}(\ell) + \phi_\ell - \frac{1}{2}\tilde{\eta}\sum_{\ell'<\ell} G(\ell,\ell')\delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)}\, A(\ell') \quad (8.51)$$

with the zero-average Gaussian random fields $\{\phi\}$, characterized by

$$\langle\phi_\ell\phi_{\ell'}\rangle_{\{\phi|A,Z\}} = \frac{1}{2}[1+C(\ell,\ell')]\,\delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)}. \quad (8.52)$$

Equation (8.51) is to be interpreted as follows. For every realization $\{Z\}$ of the fake history 'path' one iterates (8.51) to find successive values of the bid upon generating the zero-average Gaussian random variables ϕ_ℓ with statistics (8.52) (which depend, in turn, on the most recent bid realizations). The final result is averaged over the random fake history paths $\{Z\}$.

Let us now summarize the structure of the present theory describing the MG in the limit $N\to\infty$, and indicate the similarities and the differences with the previous theory describing the on-line MG with fake market history:

similarities between the theory of real and fake history MGs:

- The MG with real history is described again by the effective single-agent equation (8.47), from which the usual order dynamic order parameters $\{C,G\}$ are to be solved self-consistently via (8.48).
- The scaling with N of the characteristic times in the process with history is identical to that of the model without history, provided we avoid highly biased global bid initializations (in the latter case the process with history would act faster by a factor $\sqrt{N}$).

differences between the theory of real and fake history MGs:

- The differences between real and fake history are in the retarded self-interaction kernel R and the effective noise covariance kernel Σ of the single-agent equation.

Without history, $\{R, \Sigma\}$ were found as explicit functions of $\{C, G\}$. With history they are to be solved from an effective equation (8.51) for the stochastically evolving global bid.

- Thus, whereas without history we had a single non-Markovian effective agent equation, with history we have two non-Markovian effective equations, one describing the evolution of an effective agent and one describing an effective global bid.

the effective global bid process:

- The global bid process (8.51) is in itself independent of the stochastic effective single-trader process (8.47). The two are linked only via the (time-dependent) order parameters occurring in their definitions.
- At each stage ℓ in the process (8.51), the bid $A(\ell)$ is coupled directly only to those bids in the past at times ℓ' with *identical realization of the M-bit history string*. In addition, according to (8.52) only those effective global bid noise variables ϕ_ℓ are correlated which correspond to times ℓ with identical realizations of the M-bit history string.

As always, the link between the non-disordered effective single-agent process (8.47) and the original disordered N-trader process (8.1) and (8.2) is as follows. If we calculate for the effective agent (8.47) the path probability $P[q(0), q(1), q(2), \ldots]$, then this object is to be interpreted for the original N-agent process as

$$P[q(0), q(1), q(2), \cdots] \\ = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle\, \delta[q(0) - q_i(0)]\, \delta[q(1) - q_i(1)]\, \delta[q(2) - q_i(2)] \;\cdots\; \rangle}.$$

In particular, at any given time t one must interpret the marginal probability density $P_t(q) = \langle \delta[q - q(t)] \rangle_\star$ to find value q at time t in terms of the original MG system as $P_t(q) = \lim_{N\to\infty} N^{-1} \sum_i \overline{\langle \delta[q - q_i(t)] \rangle}$. Mathematically, the differences between the two 'fake history' definitions (2.3) and (2.4) (i.e. consistent versus inconsistent) are seen to be limited only to the details of the zero-average Gaussian variables $\{Z\}$ in (8.4), and the associated averaging process $\langle \ldots \rangle_{\{Z\}}$.

8.4 Recovering the fake history limit

It will be helpful for our understanding of the technical details of the $N \to \infty$ limit in equations (8.49) and (8.50) to first return to the simplest case where we know what the outcome should be, being $\zeta = 1$, i.e. fake history strings of the inconsistent type (2.4).

This is the model which we have solved in Chapter 5. We will thereby also *passant* re-confirm the correctness of the assumed scaling $\delta_N = \tilde{\eta}/2p$.

8.4.1 Consequences for the theory

For $\zeta = 1$ we observe in equations (8.4) and (8.7) that both $\boldsymbol{\lambda}(\ldots)$ and $\overline{W}[\ldots]$ lose their dependence on the true path $\{A\}$ of the global bid, and reduce to

$$\boldsymbol{\lambda}(\ell, Z) = \begin{pmatrix} \mathrm{sgn}[Z(\ell,1)] \\ \vdots \\ \mathrm{sgn}[Z(\ell,M)] \end{pmatrix} \qquad \overline{W}[\ell,\ell';Z] = \delta_{\boldsymbol{\lambda}(\ell,Z),\boldsymbol{\lambda}(\ell',Z)}. \tag{8.53}$$

The role of the Gaussian variables $\{Z\}$ has thereby been reduced to determining the statistics of the symmetric random matrix $\mathcal{B}$ with entries $\mathcal{B}_{\ell\ell'} = \overline{W}[\ell,\ell';Z]$:

$$\mathcal{P}[\mathcal{B}] = \left\langle \prod_{\ell,\ell'} \delta\Big[\mathcal{B}_{\ell\ell'} - \prod_{\lambda=1}^{\log_2(p)} \theta[Z(\ell,\lambda)Z(\ell',\lambda)]\Big] \right\rangle_{\{Z\}} \tag{8.54}$$

with $p = 2^M = \alpha N$. The relevant properties of these matrices are derived in Appendix C. The partial decoupling of the paths $\{A\}$ and $\{Z\}$ implies that our expressions for the kernels R and Σ simplify to

$$R(t,t') = \frac{\delta}{\delta A_{\mathrm{e}}(t')} \lim_{\delta_N \to 0} \left\langle \mathcal{B}_{\ell\ell'} \left\langle A(\ell) \right\rangle_{\{A|\mathcal{B}\}} \right\rangle_{\mathcal{B}} \Big|_{\ell = t/\delta_N, \ell' = t'/\delta_N}, \tag{8.55}$$

$$\Sigma(t,t') = \tilde{\eta} \lim_{\delta_N \to 0} \frac{1}{\delta_N} \left\langle \mathcal{B}_{\ell\ell'} \left\langle A(\ell)A(\ell') \right\rangle_{\{A|\mathcal{B}\}} \right\rangle_{\mathcal{B}} \Big|_{\ell = t/\delta_N, \ell' = t'/\delta_N}. \tag{8.56}$$

Since the bid evolution process (8.39) is now linear in $\{A\}$, and involves only $\{A\}$-independent zero-average Gaussian fields ϕ_ℓ, where $\langle \phi_\ell \phi_{\ell'} \rangle_{\{\phi|\mathcal{B}\}} = \frac{1}{2}\mathcal{B}_{\ell\ell'}[1 + C(\ell,\ell')]$, it is easily solved for any given realization of the random matrix $\mathcal{B}$:

$$A(\ell) = A_{\mathrm{e}}(\ell) + \phi_\ell + \sum_{r>0} \left(-\frac{\tilde{\eta}}{2}\right)^r \sum_{k<\ell} [(G\mathcal{B})^r]_{\ell k} \left[A_{\mathrm{e}}(k) + \phi_k\right] \tag{8.57}$$

in which $G\mathcal{B}$ denotes the matrix with entries $(G\mathcal{B})_{\ell\ell'} = G(\ell,\ell')\mathcal{B}_{\ell\ell'}$ (i.e. it is defined in terms of component multiplication rather than matrix multiplication). To make a comparison with the results of Chapter 5, we must send the external bid perturbations $A_{\mathrm{e}}(\ell)$ to zero after they have served to generate the response function R.

8.4.2 The retarded self-interaction kernel

It turns out that we can evaluate equation (8.55) using only expression (8.57), the causality of the response function, and the formulas (C.3) and (C.4) of Appendix C. These give, with $\delta/\delta A_{\rm e}(\ell) = \delta_N{}^{-1}\partial/\partial A(\ell)$:

$$
\begin{aligned}
R(\ell,\ell') &= \lim_{A_{\rm e}\to 0}\lim_{\delta_N\to 0}\frac{\partial}{\partial A_{\rm e}(\ell')}\frac{1}{\delta_N}\int {\rm d}\mathcal{B}\,\mathcal{P}[\mathcal{B}]\;\mathcal{B}_{\ell\ell'}\;\langle A(\ell)\rangle_{\{\phi|\mathcal{B}\}}\\
&= \lim_{\delta_N\to 0}\frac{1}{\delta_N}\sum_{r\geq 0}\left(-\frac{\tilde{\eta}}{2}\right)^r\int {\rm d}\mathcal{B}\,\mathcal{P}[\mathcal{B}]\;\mathcal{B}_{\ell\ell'}[(G\mathcal{B})^r]_{\ell\ell'}\\
&= \lim_{\delta_N\to 0}\frac{1}{\delta_N}\Bigg\{\delta_{\ell\ell'}\int {\rm d}\mathcal{B}\,\mathcal{P}[\mathcal{B}]\;\mathcal{B}_{\ell\ell}-\frac{\tilde{\eta}}{2}G(\ell,\ell')\int {\rm d}\mathcal{B}\,\mathcal{P}[\mathcal{B}]\;\mathcal{B}_{\ell\ell'}\mathcal{B}_{\ell'\ell}\\
&\quad+\sum_{r>1}\left(-\frac{\tilde{\eta}}{2}\right)^r\sum_{s_2>s_3>\cdots>s_r}G(\ell,s_2)G(s_2,s_3)\cdots G(s_r,\ell')\\
&\quad\times\int {\rm d}\mathcal{B}\,\mathcal{P}[\mathcal{B}]\;\mathcal{B}_{\ell,s_2}\mathcal{B}_{s_2,s_3}\cdots\mathcal{B}_{s_r,\ell'}\mathcal{B}_{\ell'\ell}\Bigg\}\\
&= \lim_{\delta_N\to 0}\frac{1}{\delta_N}\Bigg\{\delta_{\ell\ell'}-\delta_N G(\ell,\ell')\\
&\quad+\sum_{r>1}(-\delta_N)^r\sum_{s_2>s_3>\cdots>s_r}G(\ell,s_2)G(s_2,s_3)\cdots G(s_r,\ell')\Bigg\}.
\end{aligned}
\tag{8.58}
$$

We observe in (8.58), in view of $\delta_{\ell\ell'}\to\delta_N\delta(t-t')$ in the continuous time limit $N\to\infty$, that the canonical scaling of time (modulo $\mathcal{O}(1)$ factors) is indeed $\delta_N=\tilde{\eta}/2p$. We now find exactly expression (5.54), as derived earlier in Chapter 5 for the on-line 'fake history' MG

$$
R(t,t')=\delta(t-t')+\sum_{r>0}(-1)^r G^r(t,t')=[\mathbf{1}+G]^{-1}(t,t'). \tag{8.59}
$$

Had we chosen an alternative scaling with N of δ_N, we would have found in the limit $\delta_N\to 0$ either the trivial result $R=0$, or an ill-defined expression.

8.4.3 The noise covariance kernel

Next we turn to expression (8.50) for the effective agent's noise covariances, with $A_{\rm e}=0$. The equivalence of our present expression and that in Chapter 5 will be more transparent upon renaming $(\ell,\ell')\to(s_0,s_0')$ and $D(k,k')=1+C(k,k')$:

$$
\Sigma(s_0,s_0')=\lim_{\delta_N\to 0}\frac{\tilde{\eta}}{\delta_N}\int {\rm d}\mathcal{B}\,\mathcal{P}[\mathcal{B}]\;\mathcal{B}_{s_0s_0'}\;\langle A(s_0)A(s_0')\rangle_{\{\phi|\mathcal{B}\}}
$$

$$
\begin{aligned}
&= \lim_{\delta_N \to 0} \frac{\tilde{\eta}}{2\delta_N} \sum_{r,r' \geq 0} \left(-\frac{\tilde{\eta}}{2}\right)^{r+r'} \sum_{k,k'} D(k,k') \\
&\quad \times \int \mathrm{d}\mathcal{B}\, \mathcal{P}[\mathcal{B}]\, [(G\mathcal{B})^r]_{s_0 k} \mathcal{B}_{kk'} \mathcal{B}_{s_0 s_0'} [(G\mathcal{B})^{r'}]_{s_0' k'} \\
&= \lim_{\delta_N \to 0} \frac{\tilde{\eta}}{2\delta_N} \sum_{r,r' \geq 0} \left(-\frac{\tilde{\eta}}{2}\right)^{r+r'} \sum_{s_1,\ldots,s_r} \sum_{s_1',\ldots,s_{r'}'} D(s_r, s_{r'}') \\
&\quad \times G(s_0,s_1),\cdots,G(s_{r-1},s_r)\; G(s_0',s_1'),\cdots,G(s_{r'-1}',s_{r'}') \\
&\quad \times \langle (\mathcal{B}_{s_0 s_1},\cdots,\mathcal{B}_{s_{r-1}s_r}) \mathcal{B}_{s_r s_{r'}'} (\mathcal{B}_{s_0' s_1'},\cdots,\mathcal{B}_{s_{r-1}' s_r'}) \mathcal{B}_{s_0 s_0'} \rangle_{\mathcal{B}}
\end{aligned}
\tag{8.60}
$$

with the proviso that in the case where $r = 0$ we must interpret the sums as $\sum_{s_1,\ldots,s_2} \to 1$, $G(s_0,s_1),\ldots,G(s_{r-1},s_r) \to 1$ and $\mathcal{B}_{s_0 s_1},\ldots,\mathcal{B}_{s_{r-1}s_r} \to 1$ (and similarly when $r' = 0$). Since the kernel $\Sigma(s_0,s_0')$ is by definition symmetric, we may without loss of generality choose $s_0' \geq s_0$. Let us work out the averages over the ensemble $\mathcal{P}[\mathcal{B}]$ in expression (8.60) further, using the shorthand $\overline{\delta}_{ij} = 1 - \delta_{ij}$ and the results of Appendix C. Dependent on the whether any or both of the indices (r,r') are zero, we have to evaluate the following averages

- $r = r' = 0$:
  Here the average of the last line in (8.60) reduces to (C.3):

$$
\langle\cdots\rangle_{\mathcal{B}} = \langle \mathcal{B}^2_{s_0 s_0'} \rangle = \langle \mathcal{B}_{s_0 s_0'} \rangle = \delta_{s_0 s_0'} + \frac{1}{p} \overline{\delta}_{s_0 s_0'}.
\tag{8.61}
$$

- $r' = 0,\ r > 0$:
  Here the average in (8.60) reduces to two terms (representing the cases $s_0 = s_0'$ versus $s_0 < s_0'$) which are both of the form (C.4):

$$
\langle\cdots\rangle_{\mathcal{B}} = \langle (\mathcal{B}_{s_0 s_1},\ldots,\mathcal{B}_{s_{r-1}s_r}) \mathcal{B}_{s_r s_0'} \mathcal{B}_{s_0 s_0'} \rangle = p^{-r} \delta_{s_0 s_0'} + p^{-r-1} \overline{\delta}_{s_0 s_0'},
\tag{8.62}
$$

  where we have used the property $\mathcal{B}_{kk} = 1$, for any k. The situation $r = 0$, $r' > 0$ is clearly equivalent.

- $r, r' > 0$:
  Now the relevant average reduces to that of (C.6):

$$
\begin{aligned}
\langle\ldots\rangle_{\mathcal{B}} &= \langle [\mathcal{B}_{s_0 s_1},\ldots,\mathcal{B}_{s_{r-1}s_r}] \mathcal{B}_{s_r s_{r'}'} [\mathcal{B}_{s_0' s_1'},\ldots,\mathcal{B}_{s_{r'-1}' s_{r'}'}] \mathcal{B}_{s_0 s_0'} \rangle \\
&= p^{\sum_{i=0}^{r} \sum_{j=0}^{r'} \delta_{s_i s_j'} - r - r' - 1}.
\end{aligned}
\tag{8.63}
$$

We observe that expression (8.63) reduces to those derived for the simpler cases where r or r' is zero (or both), so it is in fact correct for any combination (r, r'). We may therefore insert (8.63) into (8.60), and obtain

$$\begin{aligned}\Sigma(s_0, s_0') &= \lim_{\delta_N\to 0} \sum_{r,r'\geq 0} (-\delta_N)^{r+r'} \sum_{s_1\ldots s_r} \sum_{s_1'\ldots s_{r'}'} D(s_r, s_{r'}') \prod_{i=0}^{r}\prod_{j=0}^{r'} \left[1 + (p-1)\delta_{s_i s_j'}\right] \\ &\quad \times G(s_0, s_1), \ldots, G(s_{r-1}, s_r)\, G(s_0', s_1'), \ldots, G(s_{r'-1}', s_{r'}') \\ &= \lim_{N\to\infty} \sum_{r,r'\geq 0} (-\delta_N)^{r+r'} \sum_{s_1,\ldots,s_r} \sum_{s_1'\ldots s_{r'}'} D(s_r, s_{r'}') \\ &\quad \times \prod_{i=0}^{r}\prod_{j=0}^{r'} \left[1 + \frac{\tilde{\eta}}{2\delta_N}\delta_{s_i s_j'}[1 - \mathcal{O}(\delta_N)]\right] \\ &\quad \times G(s_0, s_1), \ldots, G(s_{r-1}, s_r)\, G(s_0', s_1'), \ldots, G(s_{r'-1}', s_{r'}') \\ &= \sum_{r,r'\geq 0} (-1)^{r+r'} \int_0^\infty \mathrm{d}s_1, \ldots, \mathrm{d}s_r \mathrm{d}s_1', \ldots, \mathrm{d}s_{r'}' \prod_{i=0}^{r}\prod_{j=0}^{r'} \left[1 + \frac{1}{2}\tilde{\eta}\delta[s_i - s_j']\right] \\ &\quad \times G(s_0, s_1), \ldots, G(s_{r-1}, s_r)\, G(s_0', s_1'), \ldots, G(s_{r'-1}', s_{r'}') \\ &\quad \times [1 + C(s_r, s_{r'}')]. \end{aligned} \tag{8.64}$$

This is indeed exactly equation (5.55) of Chapter 5, as it should be.

8.5 The role of history statistics

We continue with our analysis of the full MG with history, and next show that all the complications induced by having real market history can ultimately be concentrated in the statistics of the M-bit memory strings $\boldsymbol{\lambda}$ of (8.4). More specifically, the core objects in the theory will turn out to be the following functions, which measure the joint probability to find identical histories in the effective global bid process (8.51) at k specified times $\{\ell_1, \ldots, \ell_k\}$, relative to the probability p^{-k} for this to happen in the case of randomly drawn fake histories and non-identical times:

$$\Delta_k(\ell_1, \ldots, \ell_k) = p^{k-1} \sum_{\boldsymbol{\lambda}} \left\langle\!\!\left\langle \prod_{i=1}^{k} \delta_{\boldsymbol{\lambda}, \boldsymbol{\lambda}(\ell_i, A, Z)} \right\rangle\!\!\right\rangle_{\{A,Z\}}. \tag{8.65}$$

Here we have abbreviated $\sum_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\lambda}\in\{-1,1\}^M}$, with $2^M = p = \alpha N$. Thus, for any value of k one recovers in the random history limit and for non-identical times simply $\lim_{\zeta\to 1} \Delta_k(\ldots) = 1$. For $k = 1$ one has $\Delta_1(\ell) = \sum_{\boldsymbol{\lambda}} \langle\!\langle \delta_{\boldsymbol{\lambda}, \boldsymbol{\lambda}(\ell, A, Z)} \rangle\!\rangle_{\{A,Z\}} = 1$, for any ζ. In contrast, for arbitrary ζ (i.e. when we allow for real histories) and for $k > 1$ the functions (8.65) will be highly non-trivial.

8.5.1 Reduction of the kernels $\{\boldsymbol{R}, \boldsymbol{\Sigma}\}$

We will try to follow as closely as possible the steps which we took earlier in order to recover the $\zeta = 1$ equations (8.59) and (8.64). Thus we rewrite the global bid equation (8.51) as

$$\sum_{\ell' \leq \ell} \Big\{ \delta_{\ell\ell'} + \frac{1}{2}\tilde{\eta} G(\ell,\ell')\delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)} \Big\} A(\ell') = A_{\mathrm{e}}(\ell) + \phi_\ell$$

and we formally invert the operator on the left-hand side, using $\delta_N = \tilde{\eta}/2p$:

$$A(\ell) = A_{\mathrm{e}}(\ell) + \phi_\ell + \sum_{r>0}\left(-\frac{\tilde{\eta}}{2}\right)^r \sum_{\ell_1,\ldots,\ell_r} G(\ell,\ell_1)G(\ell_1,\ell_2),\ldots,G(\ell_{r-1},\ell_r)$$
$$\times \Big[\prod_{i=1}^{r} \delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell_i,A,Z)}\Big]\,[A_{\mathrm{e}}(\ell_r) + \phi_{\ell_r}]. \tag{8.66}$$

Expression (8.66), although correct, does not constitute a solution of (8.51), since the bids $\{A(s)\}$ also occur inside the history strings $\boldsymbol{\lambda}(\ell', A, Z)$ at the right-hand side. We now insert (8.66) first into the kernel (8.49), and consider for now only infinitesimally small external bid perturbations A_{e}, so that we need not worry about the indirect effects on $A(\ell)$ of these perturbations via the history strings $\boldsymbol{\lambda}(s, A, Z)$:

$$R(t,t') = \delta(t-t') + \lim_{\delta_N\to 0}\Bigg\{\sum_{r>0}(-\delta_N)^{r-1}\sum_{\ell_1,\ldots,\ell_{r-1}} G(\ell,\ell_1)G(\ell_1,\ell_2),\ldots,G(\ell_{r-1},\ell')$$
$$\times\, p^r \Big\langle\Big\langle \delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell',A,Z)} \prod_{i=1}^{r-1}\delta_{\boldsymbol{\lambda}(\ell,A,Z),\boldsymbol{\lambda}(\ell_i,A,Z)}\Big\rangle\Big\rangle_{\{A,Z\}}\Bigg\}\Bigg|_{\ell=\frac{t}{\delta_N},\,\ell'=\frac{t'}{\delta_N}}$$
$$= \delta(t-t') + \lim_{\delta_N\to 0}\Bigg\{\sum_{r>0}(-\delta_N)^{r-1}\sum_{\ell_1,\ldots,\ell_{r-1}} G(\ell_0,\ell_1),\ldots,G(\ell_{r-1},\ell_r)$$
$$\times\, p^r \sum_{\boldsymbol{\lambda}}\Big\langle\Big\langle \prod_{i=0}^{r}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_i,A,Z)}\Big\rangle\Big\rangle_{\{A,Z\}}\Bigg\}\Bigg|_{\ell_0=\frac{t}{\delta_N},\,\ell_r=\frac{t'}{\delta_N}}$$
$$= \delta(t-t') + \lim_{\delta_N\to 0}\Bigg\{\sum_{r>0}(-\delta_N)^{r-1}\sum_{\ell_1,\ldots,\ell_{r-1}} G(\ell_0,\ell_1),\ldots,G(\ell_{r-1},\ell_r)$$

$$\times\ \Delta_{r+1}(\ell_0,\ldots,\ell_r)\Big\}\Big|_{\ell_0=\frac{t}{\delta_N},\ell_r=\frac{t'}{\delta_N}}. \tag{8.67}$$

Similarly we can insert (8.66) into (8.50), again with $A_{\mathrm{e}}\to 0$, and find

$$\begin{aligned}
\Sigma(t,t') &= \tilde{\eta}\lim_{\delta_N\to 0}\frac{1}{\delta_N}\Bigg\{\sum_{r,r'\geq 0}(-\delta_N)^{r+r'}\sum_{\ell_1,\ldots,\ell_r}G(\ell_0,\ell_1),\ldots,G(\ell_{r-1},\ell_r)\\
&\quad\times\sum_{\ell'_1,\ldots,\ell'_r}G(\ell'_0,\ell'_1),\ldots,G(\ell'_{r-1},\ell'_r)\,p^{r+r'}\Big\langle\!\Big\langle\ \langle\phi_{\ell_r}\phi_{\ell'_{r'}}\rangle_{\{\phi|A,Z\}}\\
&\quad\times\ \Big[\prod_{i=1}^{r}\delta_{\boldsymbol{\lambda}(\ell_0,A,Z),\boldsymbol{\lambda}(\ell_i,A,Z)}\Big]\Big[\prod_{j=1}^{r'}\delta_{\boldsymbol{\lambda}(\ell'_0,A,Z),\boldsymbol{\lambda}(\ell'_j,A,Z)}\Big]\ \Big\rangle\!\Big\rangle_{\{A,Z\}}\Bigg\}\Bigg|_{\ell_0=\frac{t}{\delta_N},\ell'_0=\frac{t'}{\delta_N}}\\
&= \lim_{\delta_N\to 0}\Bigg\{\sum_{r,r'\geq 0}(-\delta_N)^{r+r'}\sum_{\ell_1,\ldots,\ell_r}G(\ell_0,\ell_1),\ldots,G(\ell_{r-1},\ell_r)\\
&\quad\times\sum_{\ell'_1,\ldots,\ell'_r}G(\ell'_0,\ell'_1),\ldots,G(\ell'_{r-1},\ell'_r)\,\big[1+C(\ell_r,\ell'_{r'})\big]\\
&\quad\times p^{r+r'+1}\sum_{\boldsymbol{\lambda}}\Big\langle\!\Big\langle\Big[\prod_{i=0}^{r}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_i,A,Z)}\Big]\Big[\prod_{j=0}^{r'}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell'_j,A,Z)}\Big]\Big\rangle\!\Big\rangle_{\{A,Z\}}\Bigg\}\Bigg|_{\ell_0=\frac{t}{\delta_N},\ell'_0=\frac{t'}{\delta_N}}\\
&= \lim_{\delta_N\to 0}\Bigg\{\sum_{r,r'\geq 0}(-\delta_N)^{r+r'}\sum_{\ell_1,\ldots,\ell_r}G(\ell_0,\ell_1),\ldots,G(\ell_{r-1},\ell_r)\\
&\quad\times\sum_{\ell'_1,\ldots,\ell'_r}G(\ell'_0,\ell'_1),\ldots,G(\ell'_{r-1},\ell'_r)\,\big[1+C(\ell_r,\ell'_{r'})\big]\\
&\quad\times\ \Delta_{r+r'+2}(\ell_0,\ldots,\ell_r,\ell'_0,\ldots,\ell'_{r'})\Bigg\}\Bigg|_{\ell_0=\frac{t}{\delta_N},\ell'_0=\frac{t'}{\delta_N}}.
\end{aligned} \tag{8.68}$$

We see that the limits $\delta_N\to 0$ in (8.67) and (8.68) are both well-defined, since each time summation will combine with a factor δ_N to generate an integral, whereas pairwise identical times in (8.68)[48] will leave a 'bare' factor δ_N but will also cause the function $\Delta_{r+r'+2}(\ldots)$ to gain a factor $p=\tilde{\eta}/2\delta_N$ in compensation.

Since the effective single-agent process (8.47) is linked to the effective global bid process (8.51) only via the two kernels $\{R,\Sigma\}$, we must conclude from the above results (8.67) and (8.68) (which are still exact at all times and for all values of the MG's control parameters) that the effects of having true market history are concentrated solely in the resulting history statistics as described by the functions (8.65). More

[48] In expression (8.67), in contrast, the occurrence of identical times in the summations is ruled out by the causality of the response function.

specifically, it follows that there is no need for us to solve the global bid process (8.51) beyond knowing the history statistics which it generates.

8.5.2 TTI stationary states

In fully ergodic and TTI states without anomalous response, we could in previous MG versions find exact closed equations for persistent order parameters without having to solve for the kernels $\{C, G\}$ in full, and even locate phase transitions exactly. This suggests that the same may be true for MGs with true history, and that also here we can possibly find such exact results without knowing the full history statistics (8.65). Thus we make the standard TTI ansatz for the kernels in (8.47) and for the correlation- and response functions:

$$C(t,t') = C(t-t'), \quad G(t,t') = G(t-t'),$$
$$R(t,t') = R(t-t'), \quad \Sigma(t,t') = \Sigma(t-t'),$$

with $\chi = \int_0^\infty \mathrm{d}t\, G(t)$ finite. It turns out that several of the familiar relations between persistent observables in TTI stationary states of the present non-Markovian MG process, if such states again exist, can be established on the basis of (8.47) alone. If we follow the notational conventions adopted previously and write time averages as $\overline{f} = \lim_{\tau\to\infty} \tau^{-1} \int_0^\tau \mathrm{d}t\, f(t)$, we may write the time average of (8.47) as

$$\overline{\mathrm{d}q/\mathrm{d}t} = \overline{\tilde{\theta}} - \alpha\chi_R\, \overline{\sigma} + \sqrt{\alpha}\, \overline{\eta}, \tag{8.69}$$

with $\chi_R = \int_0^\infty \mathrm{d}t\, R(t)$. We may now define the familiar effective agent trajectories corresponding to fickle versus frozen agents as those with either $\overline{\mathrm{d}q/\mathrm{d}t} = 0$ or $\overline{\mathrm{d}q/\mathrm{d}t} \neq 0$, respectively. In the case of frozen agents, consistency demands that $\mathrm{sgn}[\overline{\sigma}] = \mathrm{sgn}[\overline{\mathrm{d}q/\mathrm{d}t}]$. It then follows from (8.69) that the (at least as long as $\chi_R > 0$, complementary, and mutually exclusive) conditions for having a 'fickle' or a 'frozen' solution can be written as follows

$$\text{fickle}: \quad |\overline{\theta} + \sqrt{\alpha}\overline{\eta}| \leq \alpha\chi_R\sigma[\infty], \quad \overline{\sigma} = \frac{\overline{\tilde{\theta}} + \sqrt{\alpha}\,\overline{\eta}}{\alpha\chi_R}, \tag{8.70}$$

$$\text{frozen}: \quad |\overline{\theta} + \sqrt{\alpha}\overline{\eta}| > \alpha\chi_R\sigma[\infty], \quad \overline{\sigma} = \sigma[\infty].\mathrm{sgn}\left[\frac{\overline{\tilde{\theta}} + \sqrt{\alpha}\,\overline{\eta}}{\alpha\chi_R}\right]. \tag{8.71}$$

Which of the solution types (8.70) or (8.71) we will find is controlled by the realization of the persistent noise term $\overline{\eta}$, which is a frozen Gaussian variable with zero expectation value and with variance given by

$$S_0^2 = \langle \overline{\eta}^2 \rangle_\star = \lim_{\tau\to\infty} \frac{1}{\tau^2} \int_0^\tau \mathrm{d}t\mathrm{d}t' \; \Sigma(t,t') = \Sigma(\infty). \tag{8.72}$$

We may now proceed as in Chapter 5 towards the calculation of the usual persistent order parameters ϕ, χ and c, where ϕ denotes the fraction of frozen agents in the stationary state, where $\chi = \int_0^\infty \mathrm{d}t \; G(t)$, and with

$$c = \lim_{t\to\infty} C(t) = \lim_{\tau\to\infty} \frac{1}{\tau^2} \int_0^\tau \mathrm{d}t\mathrm{d}t' \; \langle \sigma[q(t)]\sigma[q(t')] \rangle_\star = \langle \overline{\sigma}^2 \rangle_\star. \tag{8.73}$$

Upon introducing the shorthand $u = \sqrt{\alpha}\chi_{_R}\sigma[\infty]/S_0\sqrt{2}$, and upon using the conditions and relations (8.70) and (8.71), we find in the limit $\tilde{\theta} \to 0$ of vanishing external fields:

$$\begin{aligned}
\phi &= \int \frac{\mathrm{d}\overline{\eta}}{S_0\sqrt{2\pi}} \; \mathrm{e}^{-\frac{1}{2}\overline{\eta}^2/S_0^2} \theta\Big[|\overline{\eta}| - \sqrt{\alpha}\chi_{_R}\sigma[\infty]\Big] \\
&= 1 - \mathrm{Erf}[u],
\end{aligned} \tag{8.74}$$

$$\begin{aligned}
c &= \int \frac{\mathrm{d}\overline{\eta}}{S_0\sqrt{2\pi}} \; \mathrm{e}^{-\frac{1}{2}\overline{\eta}^2/S_0^2} \Big\{ \theta\Big[|\overline{\eta}| - \sqrt{\alpha}\chi_{_R}\sigma[\infty]\Big]\sigma^2[\infty] \\
&\qquad + \theta\Big[\sqrt{\alpha}\chi_{_R}\sigma[\infty] - |\overline{\eta}|\Big] \frac{\overline{\eta}^2}{\alpha\chi_{_R}^2} \Big\} \\
&= \sigma^2[\infty]\Big\{ 1 - \mathrm{Erf}[u] + \frac{1}{2u^2}\mathrm{Erf}[u] - \frac{1}{u\sqrt{\pi}}\mathrm{e}^{-u^2} \Big\},
\end{aligned} \tag{8.75}$$

$$\begin{aligned}
\chi &= \int \frac{\mathrm{d}\overline{\eta}}{S_0\sqrt{2\pi}} \; \mathrm{e}^{-\frac{1}{2}\overline{\eta}^2/S_0^2} \frac{\partial \overline{\sigma}}{\partial(\sqrt{\alpha}\overline{\eta})} \\
&= \frac{1}{\alpha\chi_{_R}} \int \frac{\mathrm{d}\overline{\eta}}{S_0\sqrt{2\pi}} \; \mathrm{e}^{-\frac{1}{2}\overline{\eta}^2/S_0^2} \theta\Big[\sqrt{\alpha}\chi_{_R}\sigma[\infty] - |\overline{\eta}|\Big] \\
&= \mathrm{Erf}[u]/\alpha\chi_{_R} = (1-\phi)/\alpha\chi_{_R}.
\end{aligned} \tag{8.76}$$

It is now clear that in order to find the TTI stationary state solution $\{\phi, c, \chi\}$ and the phase transition point (defined by $\chi \to \infty$, i.e. $\chi_{_R} = 0$), we only need to extract expressions for the two persistent scalar quantities $\chi_{_R}$ and S_0 from the stochastic overall bid process (8.51). With the help of (8.67) and (8.68), these two quantities can be written as

$$\begin{aligned}
\chi_{_R} &= \int_0^\infty \mathrm{d}t \; R(t) \\
&= 1 + \lim_{\delta_N \to 0} \Big\{ \sum_{r>0} (-\delta_N)^r \sum_{\ell_1,\ldots,\ell_r} G(\ell_1 - \ell_2)G(\ell_2 - \ell_3), \ldots, G(\ell_{r-1} - \ell_r)G(\ell_r) \\
&\qquad \times \varDelta_{r+1}(\ell_1, \ldots, \ell_r, 0) \Big\},
\end{aligned} \tag{8.77}$$

$$S_0^2 = \lim_{L\to\infty} \frac{1}{L^2} \sum_{\ell_0,\ell_0'\leq L} \lim_{\delta_N\to 0} \left\{ \sum_{r,r'\geq 0} (-\delta_N)^{r+r'} \sum_{\ell_1,\ldots,\ell_r} G(\ell_0-\ell_1),\ldots,G(\ell_{r-1}-\ell_r) \right.$$
$$\times \sum_{\ell_1',\ldots,\ell_r'} G(\ell_0'-\ell_1'),\ldots,G(\ell_{r-1}'-\ell_r')\,[1+C(\ell_r-\ell_{r'}')]$$
$$\left. \times\, \varDelta_{r+r'+2}(\ell_0,\ldots,\ell_r,\ell_0',\ldots,\ell_{r'}') \right\} \tag{8.78}$$

We also know, given the existence of the continuous time limit of the order parameter kernels, that the values $C(\ell)$ and $G(\ell)$ change only over ranges of ℓ which are of order $\mathcal{O}(N)$.

8.5.3 TTI states with short history correlation times

The calculation of the history statistics kernels (8.65) from the global bid process (8.51) is hard, but in those cases where the relevant history correlation times $L_{\rm h}$ (measured in individual iterations ℓ) in the process are much smaller than N, we can make further progress in our analysis of TTI stationary states.

We define the asymptotic frequency $\pi_{\boldsymbol{\lambda}}(A,Z)$ at which history string $\boldsymbol{\lambda}$ occurs in a given realization $\{A,Z\}$ of our process (8.51) as

$$\pi_{\boldsymbol{\lambda}}(A,Z) = \lim_{L\to\infty} \frac{1}{L} \sum_{\ell=1}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)}. \tag{8.79}$$

Obviously $\sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}}(A,Z)=1$. For $\zeta=1$ (no history) we would have $\pi_{\boldsymbol{\lambda}}=p^{-1}$ for all $\boldsymbol{\lambda}$. We may now also define the distribution $\varrho(f)$ of these asymptotic history frequencies $\pi_{\boldsymbol{\lambda}}(A,Z)$, relative to the benchmark 'no-memory' values p^{-1}, and averaged over the global bid process (8.51) in the infinite system size (i.e. continuous time) limit

$$\varrho(f) = \lim_{p\to\infty} \frac{1}{p} \sum_{\boldsymbol{\lambda}} \langle\langle \delta[f-p\pi_{\boldsymbol{\lambda}}(A,Z)]\rangle\rangle_{\{A,Z\}}. \tag{8.80}$$

Our definitions guarantee that $\int_0^\infty {\rm d}f\; f\varrho(f)=1$ for any ζ. For $\zeta=1$ we simply recover $\varrho(f)=\delta[f-1]$, i.e. all histories occur equally frequently.

At this stage we have not yet shown that the limit in (8.80) actually exists, i.e. that the history frequencies in the presence of history do indeed scale as $\pi_{\boldsymbol{\lambda}}(A,Z)=\mathcal{O}(N^{-1})$. Numerical simulations, however, confirm that this assumed scaling is indeed correct. Figure 8.1, for instance, shows the results of measuring the actual distribution $p^{-1}\sum_{\boldsymbol{\lambda}}\delta[f-p\pi_{\boldsymbol{\lambda}}(A,Z)]$ of relative history frequencies in typical simulation experiments, one carried out in the ergodic regime of the MG (large α, right panel), and one

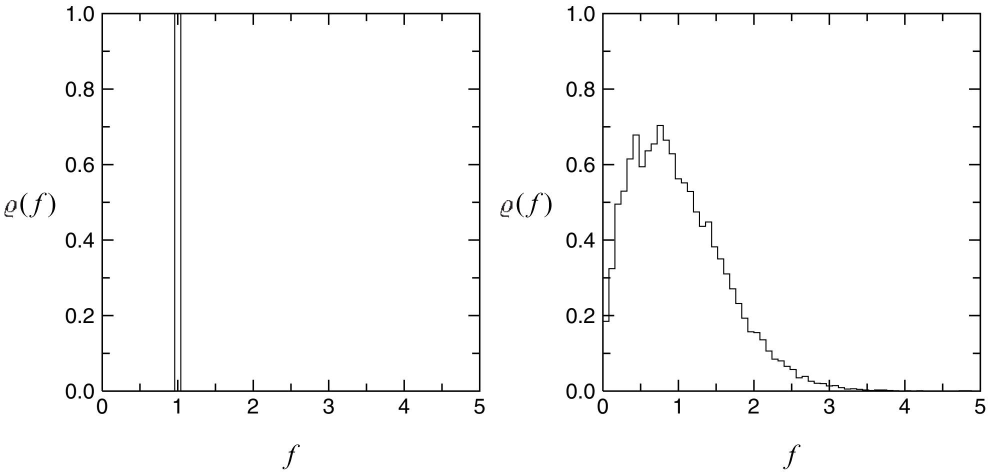


Fig. 8.1 Typical examples of history frequency distributions (8.80) as measured in simulations of the on-line MG without decision noise but with full history (i.e. $\zeta = 0$), after equilibration. Left: $\alpha = 0.125$ (in the non-ergodic regime of the MG, below α_c). Right: $\alpha = 2.0$ (in the ergodic regime, above α_c).

in the non-ergodic regime (small α, left panel). These data also indicate that above the critical point α_c the distribution of relative history frequencies for the MG with true market histories is non-trivial, and certainly does not reduce to that found for fake history version, i.e. to the delta peak $\varrho(f) = \delta[f-1]$, whereas for $\alpha < \alpha_c$ one does find $\varrho(f) = \delta[f-1]$ (within the limits of experimental accuracy).

If L_h is the characteristic history correlation time in the process (8.51), then all *finite* samples of history occurrence frequencies can be expected to approach the asymptotic value (8.79) according to

$$\frac{1}{2L} \sum_{\ell'=\ell-L}^{\ell+L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell',A,Z)} = \pi_{\boldsymbol{\lambda}}(A,Z) \left[1 + \mathcal{O}((L_h/L)^{\frac{1}{2}})\right]. \tag{8.81}$$

This implies that in expressions such as (8.77), where $G(\ell) - G(\ell') = \mathcal{O}(|\ell - \ell'|/N)$ and where only time strings $\{\ell_1, \ldots, \ell_k\}$ with mutual temporal separations of order $\mathcal{O}(N)$ will survive the limit $\delta_N \to 0$, we may choose e.g. $L = \sqrt{L_h N}$ and effectively replace

$$\Delta_{r+1}(\ell_1, \ldots, \ell_r, 0) \;\to\; p^r \sum_{\boldsymbol{\lambda}} \left[\pi_{\boldsymbol{\lambda}}[1 + \mathcal{O}(\sqrt{L_h/N})]\right]^{r+1}, \tag{8.82}$$

with

$$\pi_{\boldsymbol{\lambda}} = \langle\langle \pi_{\boldsymbol{\lambda}}(A,Z) \rangle\rangle_{\{A,Z\}}. \tag{8.83}$$

This results in

$$\chi_R = \lim_{p\to\infty} \frac{1}{p} \sum_{\boldsymbol{\lambda}} \sum_{r\geq 0} (-\chi)^r \Big[p\pi_{\boldsymbol{\lambda}}[1+\mathcal{O}(\sqrt{L_\mathrm{h}/N})]\Big]^{r+1}$$
$$= \lim_{p\to\infty} \frac{1}{p} \sum_{\boldsymbol{\lambda}} \frac{p\pi_{\boldsymbol{\lambda}}[1+\mathcal{O}(\sqrt{L_\mathrm{h}/N})]}{1+\chi\, p\pi_{\boldsymbol{\lambda}}[1+\mathcal{O}(\sqrt{L_\mathrm{h}/N})]}$$
$$= \int_0^\infty \mathrm{d}f\ \varrho(f)\ \frac{f}{1+\chi f} \tag{8.84}$$

(provided indeed $\lim_{N\to\infty} L_\mathrm{h}/N = 0$). The same simplification to an expression involving only the history frequency distribution $\varrho(f)$ can be achieved in (8.78), but there we have to be more careful in dealing with the occurrences of similar or even identical times in the argument of (8.65). We first rewrite (8.78) by transforming the iteration times according to

$$\text{for all } \ i \in \{0,\ldots,r\}: \quad \ell_i = \sum_{j=i}^{r} s_j.$$

This gives, using $\lim_{s\to\infty} G(s) = 0$ (i.e. restricting ourselves to ergodic states with normal response)

$$S_0^2 = \lim_{\delta_N\to 0} \sum_{r,r'\geq 0} (-\delta_N)^{r+r'} \sum_{s_0,\ldots,s_{r-1}>0} G(s_0),\ldots,G(s_{r-1})$$
$$\sum_{s'_0,\ldots,s'_{r'-1}>0} G(s'_0),\ldots,G(s'_{r-1})$$
$$\times \lim_{L\to\infty} \frac{1}{L^2} \sum_{s_r=0}^{L-\sum_{i=0}^{r-1} s_i} \sum_{s'_{r'}=0}^{L-\sum_{i=0}^{r'-1} s'_i} [1+C(s_r - s'_{r'})]$$
$$\times\, \Delta_{r+r'+2}(s_0+\ldots+s_r,\cdots,\ell_{r-1}+\ell_r,\ell_r,\ell'_0+\cdots+\ell_{r'},\ldots,\ell'_{r'-1}+\ell'_{r'},\ell'_{r'}).$$

Each time summation in this expression is compensated either by a factor δ_N (to combine into an integral), or limited in range by L and compensated by a corresponding prefactor L^{-1}, so that any 'pairing' where two (or more) times are close to each other (relative to the history correlation time L_h) will not survive the combined limits $\delta_N \to 0$ and $L\to\infty$. Therefore we may again replace

$$\Delta_{r+r'+2}(\ldots\ldots) \ \to\ p^{r+r'+1} \sum_{\boldsymbol{\lambda}} \Big[\pi_{\boldsymbol{\lambda}}[1+\mathcal{O}(\sqrt{L_\mathrm{h}/N})]\Big]^{r+r'+2} \tag{8.85}$$

and find, with $C(\infty) = c$:

$$S_0^2 = (1+c) \lim_{p\to\infty} \frac{1}{p} \sum_{\boldsymbol{\lambda}} \sum_{r,r'\geq 0} (-\chi)^{r+r'} \Big[p\pi_{\boldsymbol{\lambda}}[1+\mathcal{O}(\sqrt{L_\mathrm{h}/N})]\Big]^{r+r'+2}$$

$$= (1+c) \lim_{p\to\infty} \frac{1}{p} \sum_{\boldsymbol{\lambda}} \frac{\Big[p\pi_{\boldsymbol{\lambda}}[1+\mathcal{O}(\sqrt{L_{\rm h}/N})]\Big]^2}{\Big[1+\chi\, p\pi_{\boldsymbol{\lambda}}[1+\mathcal{O}(\sqrt{L_{\rm h}/N})]\Big]^2}$$

$$= (1+c) \int_0^\infty {\rm d}f\ \varrho(f)\ \frac{f^2}{(1+\chi f)^2}. \tag{8.86}$$

Since only $\chi_{_R}$ and S_0 are required to solve our effective single-agent process in TTI stationary states, we see that upon making the ansatz of short history correlation times $L_{\rm h} \ll N$ (the self-consistency of which would still have to be confirmed) the effects of history on the persistent order parameters in the MG are fully concentrated in the distribution $\varrho(f)$ of history frequencies, as defined by (8.80). Once $\varrho(f)$ has been extracted from the global bid process (8.51), the TTI persistent order parameters are apparently given by the solution of the following set of equations

$$u = \frac{\sigma[\infty]\sqrt{\alpha}\chi_{_R}}{S_0\sqrt{2}}, \quad \chi = \frac{1-\phi}{\alpha\chi_{_R}}, \quad \phi = 1 - {\rm Erf}[u], \tag{8.87}$$

$$c = \sigma^2[\infty]\Big\{1 - {\rm Erf}[u] + \frac{1}{2u^2}{\rm Erf}[u] - \frac{1}{u\sqrt{\pi}}{\rm e}^{-u^2}\Big\}, \tag{8.88}$$

$$\chi_{_R} = \int_0^\infty {\rm d}f\ \varrho(f)\ \frac{f}{1+\chi f}, \tag{8.89}$$

$$S_0^2 = (1+c)\int_0^\infty {\rm d}f\ \varrho(f)\ \frac{f^2}{(1+\chi f)^2}. \tag{8.90}$$

For $\zeta = 1$ (no history) we have $\varrho(f) = \delta[f-1]$, leading to $\chi_{_R} = (1+\chi)^{-1}$ and $S_0 = \sqrt{1+c}/(1+\chi)$, and the above equations are seen to reduce to those of Chapter 5, as they should.

8.6 Calculating the history statistics

We have seen above that, at least upon making the ansatz of short history correlation times in the MG, finding closed equations for persistent TTI order parameters boils down to the calculation of the distribution $\varrho(f)$ of relative history frequencies, as defined in (8.80). Our remaining programme of analysis is therefore: (i) find an expression for $\varrho(f)$, (ii) express this distribution in terms of the persistent TTI order parameters $\{c, \phi, \chi, \chi_{_R}, S_0\}$, and (iii) confirm retrospectively the self-consistency of assuming short history correlation times.

8.6.1 The moments of $\varrho(f)$

The distribution (8.80) is generated by the effective non-Markovian global bid process (8.51), which we cannot realistically hope to solve directly. However, we might get away with a self-consistent calculation of (8.80) which does not require solving (8.51) in full. To do so, we will focus on the moments μ_k of the distribution ϱ, from which the latter can always be recovered (if the relevant integrals exist)

$$\mu_k = \int_0^\infty \mathrm{d}f\ \varrho(f) f^k, \tag{8.91}$$

$$\varrho(f) = \int \frac{\mathrm{d}\omega}{2\pi} \mathrm{e}^{\mathrm{i}\omega f} \int_0^\infty \mathrm{d}f' \varrho(f') \mathrm{e}^{-\mathrm{i}\omega f'} = \int \frac{d\omega}{2\pi} \mathrm{e}^{\mathrm{i}\omega f} \sum_{k\geq 0} \frac{\mu_k}{k!}(-i\omega)^k. \tag{8.92}$$

We obviously have $\mu_0 = \mu_1 = 1$, for any ζ, which follows directly from the definition (8.80). In the absence of history (i.e. $\zeta = 1$) we have $\varrho(f) = \delta[f-1]$, so that $\mu_k = 1$ for all $k \geq 0$. We will rely on the sum over moments in (8.92) converging on scales of k which are independent of N. This is equivalent to stating that the limit (8.80) is well-defined, so it does not restrict us further. By simply combining the relevant definitions (8.79), (8.80) and (8.91) and (8.4), we can first of all work our way towards a more explicit but still relatively simple expression for the moments μ_k:

$$\begin{aligned}
\mu_k &= \frac{1}{p} \sum_{\boldsymbol{\lambda}} \langle\!\langle [p\pi_{\boldsymbol{\lambda}}(A,Z)]^k \rangle\!\rangle_{\{A,Z\}} \\
&= \frac{1}{p} \sum_{\boldsymbol{\lambda}} \lim_{L\to\infty} \frac{p^k}{L^k} \sum_{\ell_1,\ldots,\ell_k=1}^{L} \langle\!\langle \prod_{i=1}^{M} \prod_{j=1}^{k} \delta_{\lambda_i,\lambda_i(\ell_j,A,Z)} \rangle\!\rangle_{\{A,Z\}} \\
&= \lim_{L\to\infty} \frac{p^k}{L^k} \sum_{\ell_1,\ldots,\ell_k=1}^{L} \Big\langle\!\Big\langle \prod_{i=1}^{M} \Big\{ \frac{1}{2} \sum_{\lambda=\pm 1} \prod_{j=1}^{k} \delta_{\lambda,\lambda_i(\ell_j,A,Z)} \Big\} \Big\rangle\!\Big\rangle_{\{A,Z\}} \\
&= \lim_{L\to\infty} \frac{p^{k-1}}{L^k} \sum_{\ell_1,\ldots,\ell_k=1}^{L} \Big\langle\!\Big\langle \prod_{i=1}^{M} \Big\{ \prod_{j=1}^{k} \delta_{1,\lambda_i(\ell_j,A,Z)} + \prod_{j=1}^{k} \delta_{-1,\lambda_i(\ell_j,A,Z)} \Big\} \Big\rangle\!\Big\rangle_{\{A,Z\}}.
\end{aligned} \tag{8.93}$$

We see that the average $\langle\langle \ldots \rangle\rangle_{\{A,Z\}}$ in the last line of (8.93) gives exactly the following joint probability

$$\begin{aligned}
\langle\langle \ldots \rangle\rangle_{\{A,Z\}} = \mathrm{Prob}\Big[&\Big\{ \lambda_1(\ell_1,A,Z) = \lambda_1(\ell_2,A,Z) = \cdots = \lambda_1(\ell_k,A,Z) \Big\} \\
&\text{and } \Big\{ \lambda_2(\ell_1,A,Z) = \lambda_2(\ell_2,A,Z) = \cdots = \lambda_2(\ell_k,A,Z) \Big\} \\
&\vdots \\
&\text{and } \Big\{ \lambda_M(\ell_1,A,Z) = \lambda_M(\ell_2,A,Z) = \cdots = \lambda_M(\ell_k,A,Z) \Big\} \Big].
\end{aligned} \tag{8.94}$$

Let us define the shorthand

$$Same(i) = \Big\{\lambda_i(\ell_1, A, Z) = \lambda_i(\ell_2, A, Z) = \cdots = \lambda_i(\ell_k, A, Z)\Big\}. \qquad (8.95)$$

In words, $Same(i)$ states that the ith component of the history string takes the same value at the k specified times $\{\ell_1, \ldots, \ell_k\}$. In this new language, and given that our bid process obeys causality,[49] our statement (8.94) becomes simply

$$\begin{aligned}
\langle\langle\ldots\rangle\rangle_{\{A,Z\}} &= \mathrm{Prob}\Big[\; Same(1) \;\wedge\; Same(2) \;\wedge\; \cdots \;\wedge\; Same(M)\;\Big] \\
&= \mathrm{Prob}[Same(1) \mid Same(2) \wedge \cdots \wedge Same(M)] \\
&\quad \times \mathrm{Prob}[Same(2) \mid Same(3) \wedge \cdots \wedge Same(M)] \\
&\quad \vdots \\
&\quad \times \mathrm{Prob}[Same(M-1) \mid Same(M)] \\
&\quad \times \mathrm{Prob}[Same(M)]. \qquad (8.96)
\end{aligned}$$

Since we need not consider values of k which scale with N or L, the contributions to (8.93) from those times $\{\ell_1, \ldots, \ell_k\}$ for which there are correlations between objects at a time ℓ_r and those at another time $\ell_{r'}$ will vanish in the limit $L \to \infty$. Since we also know that we are in a TTI state, it follows that the conditional probabilities in (8.96) will not depend on the actual values $\{\ell_1, \ldots, \ell_k\}$. In the limit $L \to \infty$ we may replace

$$\mathrm{Prob}[Same(r) \mid Same(r+1) \wedge \cdots \wedge Same(M)] \;\to\; \mathcal{P}_{[k|M-r]},$$

where $\mathcal{P}_{[k|m]}$ denotes the probability to find for randomly drawn and infinitely separated times $\{\ell_1, \ldots, \ell_k\}$ that $\lambda_i(\ell_1, A, Z) = \cdots = \lambda_i(\ell_k, A, Z)$, for an index i, given that the identity holds for the indices $\{i+1, \ldots, i+m\}$ (with $\mathcal{P}_{[k|0]}$ giving this probability in the absence of conditions). This allows us to write (8.93) as

$$\mu_k = p^{k-1}\, \mathcal{P}_{[k|M-1]} \cdot \mathcal{P}_{[k|M-2]} \;\cdots\; \mathcal{P}_{[k|1]} \cdot \mathcal{P}_{[k|0]}. \qquad (8.97)$$

As a simple test one may verify (8.97) for the trivial case $\zeta = 1$ (no history in the MG). Here conditioning on the past is obviously irrelevant, so that $\mathcal{P}_{[k|m]} = \mathcal{P}_{[k|0]} = 2^{1-k}$

[49] We here use the fact that a component $\lambda_i(\ell, A, Z)$ of the history string observed by the agents at time ℓ is by construction (see definition (8.4)) referring to the overall bid at time $\ell - i$. It follows that the probability of finding a given value for $\lambda_i(\ell, A, Z)$ depends via causality only on the bids at the earlier times $\{\ell - i - 1, \ell - i - 2, \ldots\}$, hence on $\{\lambda_{i+1}(\ell, A, Z), \lambda_{i+2}(\ell, A, Z), \ldots\}$.

for all m, which indeed gives us $\mu_k = p^{k-1}2^{(1-k)M} = 1$ (as it should). In the continuous time limit $N \to \infty$ (equivalently: for $M \to \infty$, since $2^M = \alpha N$) we thus find the as yet exact formula

$$\lim_{M\to\infty} \log(\mu_k) = \lim_{M\to\infty} \sum_{r=0}^{M-1} \log\left[2^{k-1}\mathcal{P}_{[k|r]}\right]. \tag{8.98}$$

8.6.2 Reduction to history coincidence statistics

Next we have to find an expression for the probabilities $\mathcal{P}_{[k|r]}$. We know from equations (8.51) and (8.52) that the value of the overall bid at any time ℓ is only correlated with the bid value at time ℓ' if the two times (ℓ, ℓ') have *identical* history strings, i.e. if $\boldsymbol{\lambda}(\ell, A, Z) = \boldsymbol{\lambda}(\ell', A, Z)$. We know that individual histories show up during the process with probabilities of order N^{-1}. Since the likelihood of finding *recurring* histories during any number $r = \mathcal{O}(M)$ of consecutive iterations of our process is therefore vanishingly small (of order $\mathcal{O}(M/N)$) such direct correlations are of no consequence in our present calculation. It follows that the only relevant effect of conditioning in the sense of the $\mathcal{P}_{[k|r]}$ is via its biasing of histories in subsequent iterations. Although the probability of history recurrence during a time window of size $\mathcal{O}(M)$ is vanishingly small, if two (short) instances of global bid trajectories are found to have identical realizations of some of the bits of their history strings, they will nevertheless be more likely than average to have an *identical* history realization in the next time step. This is the subtle statistical effect which, together with the resulting biases in the bids which are subsequently found at times with specific histories, ultimately gives rise to the relative history frequency distributions $\varrho(f)$ as observed in Fig. 8.1.

The statement that the conditioning in $\mathcal{P}_{[k|r]}$ acts only via the joint likelihood of finding specific histories $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k\}$ at the k specified (and widely separated) times $\{\ell_1, \ldots, \ell_k\}$, translates into

$$\mathcal{P}_{[k|r]} = \sum_{\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_k} \mathcal{P}[k|\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k]\, \mathcal{P}[\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k|r] \tag{8.99}$$

Here $\mathcal{P}[k|\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k]$ denotes the likelihood to find $\boldsymbol{\lambda}(\ell_1, A, Z) = \ldots = \boldsymbol{\lambda}(\ell_k, A, Z)$, if the history strings at those k times equal $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k\}$, and $\mathcal{P}[\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k|r]$ denotes the likelihood of finding those k specific histories given that the bits of the k history strings have been identical over the r most recent iterations.[50] The probability of

[50] Here one will find that consistent and inconsistent realizations of the history noise variables $Z(\ell, i)$ are to be treated differently: in the case of consistent noise, one will always have $\lambda_i(\ell, A, Z) = \lambda_{i+1}(\ell+1, A, Z)$. This is not true for inconsistent history noise.

finding specific bid values $A(\ell)$ will in TTI states only depend on the history string $\boldsymbol{\lambda}$ associated with time ℓ. Given this history string, $A(\ell)$ is a Gaussian variable (this follows from the effective bid process (8.51)), with some average $\overline{A}_{\boldsymbol{\lambda}}$ and a variance $\sigma^2_{\boldsymbol{\lambda}}$ (which will in due course have to be calculated). Using also the fact that the $Z(\ell, i)$ were defined as zero-average Gaussian variables, with variance κ^2, we obtain

$$\begin{aligned}\mathcal{P}[k|\boldsymbol{\lambda}_1,\ldots,\boldsymbol{\lambda}_k] &= \prod_{j=1}^{k}\Big[\int DZ \int \mathrm{d}A\; P_{\boldsymbol{\lambda}_j}(A)\theta[(1-\zeta)A+\zeta Z]\Big] \\ &\quad + \prod_{j=1}^{k}\Big[\int DZ \int \mathrm{d}A\; P_{\boldsymbol{\lambda}_j}(A)\theta[-(1-\zeta)A-\zeta Z]\Big] \\ &= \prod_{j=1}^{k}\left[\frac{1}{2}+\frac{1}{2}\mathrm{Erf}\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}_j}}{\sqrt{2}\sqrt{\zeta^2\kappa^2+(1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}_j}}}\Big]\right] \\ &\quad + \prod_{j=1}^{k}\left[\frac{1}{2}-\frac{1}{2}\mathrm{Erf}\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}_j}}{\sqrt{2}\sqrt{\zeta^2\kappa^2+(1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}_j}}}\Big]\right]. \end{aligned} \tag{8.100}$$

We now write the sum over all combinations of histories in equation (8.99) in terms of a partitioning in groups, where two M-bit strings $\{\boldsymbol{\lambda}_i, \boldsymbol{\lambda}_j\}$ are in the same group if and only if they are identical. We will write $(g_1, g_2, \ldots)$ for the subset of all combinations $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k\}$ where one has one group of size g_1, a second group of size g_2, and so on.[51] Clearly $g_1 + g_2 + \cdots = k$, for all possible subsets of our partitioning. This allows us to write

$$\mathcal{P}_{[k|r]} = \sum_{(g_1,g_2,\ldots)} \delta_{k,g_1+g_2+\cdots}\mathcal{P}[k|g_1,g_2,\ldots]\;\mathcal{P}[g_1,g_2,\ldots|r]. \tag{8.101}$$

According to (8.100) the distribution $\mathcal{P}[k|g_1, g_2, \ldots]$ is of the relatively simple form $\mathcal{P}[k|g_1, g_2, \ldots] = 2^{1-k}\Phi(g_1, g_2, \ldots)$, with

$$\begin{aligned}\Phi(g_1,g_2,\ldots) &= \frac{1}{2}\prod_{j\geq 1}\left\{\sum_{\boldsymbol{\lambda}}\pi_{\boldsymbol{\lambda}}\left[1+\mathrm{Erf}\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}}}{\sqrt{2}\sqrt{\zeta^2\kappa^2+(1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}}}}\Big]\right]^{g_j}\right\} \\ &\quad +\frac{1}{2}\prod_{j\geq 1}\left\{\sum_{\boldsymbol{\lambda}}\pi_{\boldsymbol{\lambda}}\left[1-\mathrm{Erf}\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}}}{\sqrt{2}\sqrt{\zeta^2\kappa^2+(1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}}}}\Big]\right]^{g_j}\right\}. \end{aligned} \tag{8.102}$$

[51] For example: (k) denotes the subset of all combinations $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k\}$ where $\boldsymbol{\lambda}_1 = \cdots = \boldsymbol{\lambda}_k$, $(2, 1, 1, \ldots)$ is the subset of all $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k\}$ where precisely two history strings are identical, and all others are distinct.

Insertion of the representation (8.101) for $\mathcal{P}_{[k|r]}$ into (8.98) allows us to write the moments of the relative history frequencies in the following form

$$\lim_{M\to\infty}\log(\mu_k)=\lim_{M\to\infty}\sum_{r=0}^{M-1}\log\left[\sum_{(g_1,g_2,\ldots)}\delta_{k,g_1+g_2+\ldots}\Phi(g_1,g_2,\ldots)\mathcal{P}[g_1,g_2,\ldots|r]\right]. \tag{8.103}$$

It will be helpful to first assess which values of r in (8.103) can at all survive the limit $M\to\infty$. We note that whenever we have a value of r such that $M-r\to\infty$ as $M\to\infty$, the condition that the k history bits were identical over the most recent r steps still leaves a very large $\mathcal{O}(2^{M-r})$ number of compatible history strings to be found at the relevant probing times $\{\ell_1,\ldots,\ell_k\}$, so that the likelihood of finding histories coinciding in multiples $(g_1,g_2,\ldots)$ scales as

$$\mathcal{P}[g_1,g_2,\ldots|r]=\prod_{j|g_j>1}\mathcal{O}(2^{(g_j-1)(r-M)}),\quad \mathcal{P}[1,1,\ldots|r]=1+\mathcal{O}(2^{r-M}).$$

Since k is finite and $\Phi(1,1,1,\ldots)=1$, the total contribution to $\log(\mu_k)$ from all those terms where $M-r\to\infty$ as $M\to\infty$ is now found to be negligible, since for $1\ll R\ll M$ we may write

$$\begin{aligned}\sum_{r=0}^{R-1}\log\Big[\sum_{(g_1,g_2,\ldots)}&\delta_{k,g_1+g_2+\cdots}\Phi(g_1,g_2,\ldots)\mathcal{P}[g_1,g_2,\ldots|r]\Big]\\ &=\sum_{r=0}^{R-1}\log\Big[1+\mathcal{O}(2^{r-M})\Big]=\sum_{r=0}^{R-1}\mathcal{O}(2^{r-M})\\ &=\mathcal{O}(2^{-M}(2^R-1))=\mathcal{O}(2^{R-M}).\end{aligned} \tag{8.104}$$

The conclusion is that in (8.103) we need only those terms where $M-r$ is finite. This makes sense, since these latter terms represent contributions where *virtually all* past components of the history strings at the times $\{\ell_1,\ldots,\ell_k\}$ were identical, which should indeed constrain the possible histories at the times $\{\ell_1,\ldots,\ell_k\}$ most, and indeed gives the largest history coincidence rates. We consequently switch out conditioning label from the number r of previously identical components to the number $M-r$ of unconstrained components, and write

$$\mathcal{P}[g_1,g_2,\ldots|r]=\mathcal{Q}[g_1,g_2,\ldots|M-r]$$

and find (8.103) converting into the simpler form

$$\lim_{M\to\infty} \log(\mu_k) = \sum_{r\geq 1} \log\Big[\sum_{(g_1,g_2,\ldots)} \delta_{k,g_1+g_2+\ldots}\Phi(g_1,g_2,\ldots)\mathcal{Q}[g_1,g_2,\ldots|r]\Big]. \tag{8.105}$$

We are left with the task to calculate the likelihood $\mathcal{Q}[g_1, g_2, \ldots |r]$ of finding at the k distinct times $\{\ell_1, \ldots, \ell_k\}$ of our process the histories $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k\}$ to be identical in prescribed multiples of $(g_1, g_2, \ldots)$, given that the bits of the k history vectors were identical during all but r of the most recent iterations. This is still a non-trivial combinatorial problem.

At this stage of our argument we can benefit explicitly from having to consider only values of r in (8.105) which are finite (compared with M, which is sent to infinity). For each value r of the number of 'free' components, there will be only 2^r possible history strings $\boldsymbol{\lambda}$ available to be allocated to the k times $\{\ell_1, \ldots, \ell_k\}$. In principle one would have to worry about the probabilities to be assigned to each of the 2^r options. However, since we know for the full M-component history strings that their probabilities scale as $\pi_{\boldsymbol{\lambda}} = f_{\boldsymbol{\lambda}} p^{-1}$ with $f_{\boldsymbol{\lambda}} = \mathcal{O}(1)$, the effective probabilities of individual components of $\boldsymbol{\lambda} \in \{-1, 1\}$ must scale as

$$\pi_{\lambda_i} = \pi_{\boldsymbol{\lambda}}^{1/M} = \frac{1}{2} f_{\boldsymbol{\lambda}}^{1/M} = \frac{1}{2}[1 + \mathcal{O}(M^{-1})].$$

From this we deduce that for finite r we may take all 2^r allowed history strings to have equal probabilities. This turns the evaluation of $\mathcal{Q}[g_1, g_2, \ldots |r]$ into a solvable combinatorial problem. Each of k elements is given randomly one of 2^r colours (where each colour has probability 2^{-r}), and $\mathcal{Q}[g_1, g_2, \ldots |r]$ represents the likelihood of finding identical colour sets of sizes $(g_1, g_2, \ldots)$. Let us abbreviate $R = 2^r$, and write the rth term in (8.105) as $\log(H_r)$. Now, using $2^{-r(g_1+\cdots g_R)} = 2^{-rk} = R^{-k}$ we may simply write[52]

$$\lim_{M\to\infty} \log(\mu_k) = \sum_{r\geq 1} \log H_r, \tag{8.106}$$

$$H_r = \sum_{(g_1,g_2,\ldots)} \Phi(g_1, g_2, \ldots)\mathcal{Q}[g_1, g_2, \ldots |r]$$

$$= \sum_{g_1=0}^{k} \sum_{g_2=0}^{k-g_1} \sum_{g_3=0}^{k-g_1-g_2} \cdots \sum_{g_R=0}^{k-g_1-\ldots-g_{R-1}} \Phi(g_1, g_2, \ldots)\, \delta_{k,\sum_i g_i}$$

[52] One easily confirms that our expression for H_r is properly normalized. Upon choosing $\Phi(g_1, g_2, \ldots) = 1$ one can perform the summations iteratively, starting from g_R and descending down to g_1, which leads exactly to the factor R^k to combine with the R^{-k} present.

$$\times R^{-k}\binom{k}{g_1}\binom{k-g_1}{g_2}\binom{k-g_1-g_2}{g_3}\cdots\binom{k-g_1-\cdots-g_{R-1}}{g_R}. \tag{8.107}$$

8.6.3 Expansion of sign-coincidence probabilities

We have now simplified the conditional distribution $\mathcal{Q}[g_1, g_2, \ldots |r]$ of history coincidences sufficiently, and turn our attention to $\Phi(g_1, g_2, \ldots)$ as given by (8.102). If we restrict ourselves to an expansion of (8.102) in powers of the (random) bid biases $\overline{A}_{\boldsymbol{\lambda}}$ in which we retain only the leading terms, our problem simplifies further to the point where we can obtain a fully explicit expression for the moments μ_k. In Appendix D we derive the following compact relations

$$\Phi(1,1,1,\ldots) = 1, \tag{8.108}$$

$$\Phi(g_1, g_2, \ldots) = \mathrm{e}^{\frac{1}{2}\Omega\sum_{j\geq 1} g_j(g_j-1)-\frac{1}{4}\Omega^2\sum_j g_j(g_j-1)(2g_j-3)+\mathcal{O}(\Omega^3)}, \tag{8.109}$$

$$\Omega = \sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}}\, \mathrm{Erf}^2\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}}}{\sqrt{2}\sqrt{\zeta^2\kappa^2+(1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}}}}\Big]. \tag{8.110}$$

The results (8.108) and (8.109) imply that, rather than knowing the full probability distribution $\mathcal{P}[g_1, g_2, \ldots |r]$ in (8.103), we only need the (conditional) statistics of a modest number of relatively simple monomials. Expanding the exponential in (8.109) up to the relevant orders, and using $\sum_j g_j = k$ (which is always true inside (8.107)) produces

$$\begin{aligned}\Phi(g_1, g_2, \ldots) = 1 + \frac{1}{2}\Omega\Big[\sum_{j\geq 1} g_j^2 - k\Big] \\ + \frac{1}{4}\Omega^2\Big[\frac{1}{2}\sum_{ij\geq 1} g_i^2 g_j^2 - 2\sum_{j\geq 1} g_j^3 - (k-5)\sum_{j\geq 1} g_j^2 + \frac{1}{2}k^2 - 3k\Big] + \mathcal{O}(\Omega^3).\end{aligned} \tag{8.111}$$

However, since the combinatorial averaging process of (8.107) in this particular representation involves a measure which is invariant under permutations of the numbers $\{g_1, g_2, \ldots\}$, we know that the average of (8.111) is identical to that of the following simpler function (with $R = 2^r$):

$$\begin{aligned}\Phi_{\text{eff}}(g_1, g_2, \ldots) = 1 + \frac{1}{2}\Omega(Rg_1^2 - k) \\ + \frac{1}{8}\Omega^2\Big[Rg_1^4 + R(R-1)g_1^2g_2^2 - 4Rg_1^3 - 2(k-5)Rg_1^2 + k^2 - 6k\Big] \\ + \mathcal{O}(\Omega^3).\end{aligned} \tag{8.112}$$

Instead of having to use full combinatorial measure (8.107), we can therefore extract all the relevant information from the (joint) marginal distribution for the pair (g_1, g_2) only. Inserting (8.112) into (8.107) gives us

$$H_r = 1 + \frac{1}{2}\Omega\Big[RG_{2,0}^{k,R} - k\Big] + \frac{1}{8}\Omega^2\Big[RG_{4,0}^{k,R} + R(R-1)G_{2,2}^{k,R} - 4RG_{3,0}^{k,R} - 2(k-5)RG_{2,0}^{k,R} + k^2 - 6k\Big] + \mathcal{O}(\Omega^3) \tag{8.113}$$

with

$$\begin{aligned} G_{a,b}^{k,R} &= \sum_{g_1=0}^{k}\sum_{g_2=0}^{k-g_1}\sum_{g_3=0}^{k-g_1-g_2}\cdots\sum_{g_R=0}^{k-g_1-\cdots-g_{R-1}} g_1^a g_2^b\,\delta_{k,\sum_i g_i} \\ &\quad\times R^{-k}\binom{k}{g_1}\binom{k-g_1}{g_2}\binom{k-g_1-g_2}{g_3}\cdots\binom{k-g_1-\cdots-g_{R-1}}{g_R} \\ &= R^{-k}\sum_{g_1=0}^{k}\sum_{g_2=0}^{k-g_1}\binom{k}{g_1}\binom{k-g_1}{g_2}(R-2)^{k-g_1-g_2}g_1^a g_2^b. \end{aligned} \tag{8.114}$$

Those combinatorial factors $G_{a,b}^{k,R}$ which we need in order to evaluate (8.113) are calculated in Appendix E. They are found to be

$$\begin{aligned} G_{2,0}^{k,R} &= \frac{k}{R} + \frac{k(k-1)}{R^2}, \\ G_{3,0}^{k,R} &= \frac{k}{R} + \frac{3k(k-1)}{R^2} + \frac{k(k-1)(k-2)}{R^3}, \\ G_{4,0}^{k,R} &= \frac{k}{R} + \frac{7k(k-1)}{R^2} + \frac{6k(k-1)(k-2)}{R^3} + \frac{k(k-1)(k-2)(k-3)}{R^4}, \\ G_{2,2}^{k,R} &= \frac{k(k-1)}{R^2} + \frac{2k(k-1)(k-2)}{R^3} + \frac{k(k-1)(k-2)(k-3)}{R^4}. \end{aligned}$$

Insertion of these factors into (8.113), followed by restoration of the shorthand $R = 2^r$, gives us the fully explicit expression

$$H_r = 1 + \frac{1}{2}\Omega k(k-1)2^{-r} + \frac{1}{8}\Omega^2 k(k-1)(k-2)(k-3)4^{-r} + \mathcal{O}(\Omega^3). \tag{8.115}$$

This concludes our combinatorial analysis of the history string statistics. We can now give explicit formulae for the moments of the relative history frequencies, and hence also for the distribution $\varrho(f)$ itself, in the form an expansion in a parameter which controls the width of this distribution.

8.6.4 Resulting prediction for $\varrho(f)$

The result (8.115), together with the earlier relation (8.106) and the standard geometric series $\sum_{r\geq 1} x^r = x/(1-x)$ (for $|x|<1$), leads us finally to the desired expression for the moments μ_k:

$$\begin{aligned}\lim_{M\to\infty}\log(\mu_k) &= \sum_{r\geq 1}\log\Big[1+\frac{1}{2}\Omega k(k-1)2^{-r}+\frac{1}{8}\Omega^2 k(k-1)(k-2)(k-3)4^{-r}\\ &\qquad +\mathcal{O}(\Omega^3)\Big]\\ &= \frac{1}{2}\Omega k(k-1)-\frac{1}{12}\Omega^2 k(k-1)(2k-3)+\mathcal{O}(\Omega^3).\end{aligned} \tag{8.116}$$

We see that this general formula obeys $\mu_0=\mu_1=1$, as it should, and that

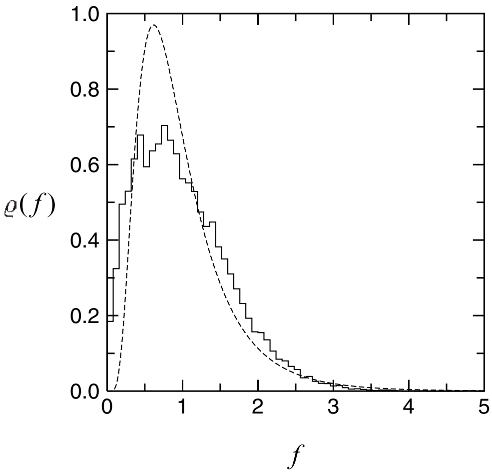
$$\lim_{M\to\infty}\mu_2 = \mathrm{e}^{\Omega-\frac{1}{6}\Omega^2+\mathcal{O}(\Omega^3)}. \tag{8.117}$$

Insertion into our earlier expression (8.92) for the relative history frequency distribution $\varrho(f)$ leads in the limit $M\to\infty$ to a formula in which, at least up to the relevant orders in Ω, the introduction of a suitable Gaussian integral allows us to carry out the summation over moments explicitly

$$\begin{aligned}\varrho(f) &= \int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}\omega f}\sum_{k\geq 0}\frac{(-\mathrm{i}\omega)^k}{k!}\mathrm{e}^{\frac{1}{2}\Omega k(k-1)-\frac{1}{12}\Omega^2 k(k-1)(2k-3)+\mathcal{O}(\Omega^3)}\\ &= \int Dz\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}\omega f}\sum_{k\geq 0}\frac{(-\mathrm{i}\omega)^k}{k!}\mathrm{e}^{zk\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}k(\Omega+\frac{1}{2}\Omega^2)}\Big[1-\frac{1}{6}\Omega^2 k^3+\cdots\Big]\\ &= \int Dz\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}\omega f}\sum_{k\geq 0}\frac{(-\mathrm{i}\omega)^k}{k!}\Big[1-\frac{1}{6}\sqrt{\Omega}\frac{\mathrm{d}^3}{\mathrm{d}z^3}+\cdots\Big]\mathrm{e}^{zk\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}k(\Omega+\frac{1}{2}\Omega^2)}\\ &= \int Dz\Big[1-\frac{1}{6}\sqrt{\Omega}\frac{\mathrm{d}^3}{\mathrm{d}z^3}+\cdots\Big]\,\delta\Big[f-\mathrm{e}^{z\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}\Big]\\ &= \int Dz\Big[1+\frac{1}{6}\sqrt{\Omega}\Big(\mathrm{e}^{\frac{1}{2}z^2}\frac{\mathrm{d}^3}{\mathrm{d}z^3}\mathrm{e}^{-\frac{1}{2}z^2}\Big)+\cdots\Big]\,\delta\Big[f-\mathrm{e}^{z\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}\Big]\\ &= \int Dz\Big[1+\frac{1}{6}\sqrt{\Omega}(3z-z^3)+\cdots\Big]\,\delta\Big[f-\mathrm{e}^{z\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}\Big].\end{aligned} \tag{8.118}$$

We may use (8.117) to express the parameter Ω in terms of the second moment μ_2, turning our expansion of the moments μ_k into an expansion in powers of $\log(\mu_2)$. Depending on whether we wish to take our expansion only to order $\mathcal{O}(\log(\mu_2))$ in the moments μ_k, or also to $\mathcal{O}(\log^2(\mu_2))$, we obtain

$$\text{to } \mathcal{O}(\log(\mu_2)):\;\; \varrho(f) = \frac{\mathrm{e}^{-\frac{1}{2}z^2(f)}}{f\sqrt{2\pi\log(\mu_2)}}, \tag{8.119}$$



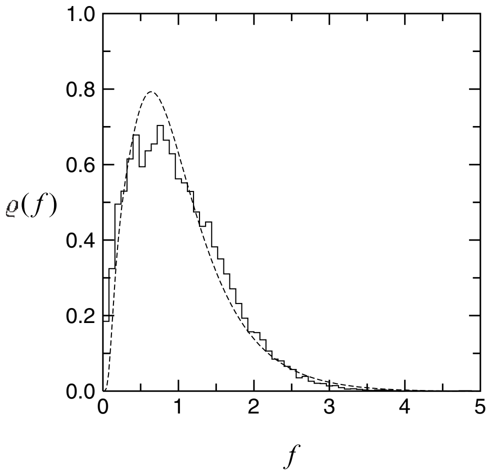


Fig. 8.2 Test of the predicted history frequency distributions (8.119) (left picture, based on expansion of the moments μ_k up to first order in the width, $\mu_k = \mathrm{e}^{\frac{1}{2}\Omega k(k-1)}$) and (8.121) (right picture, based on expansion up to second order, $\mu_k = \mathrm{e}^{\frac{1}{2}\Omega k(k-1)-\frac{1}{12}\Omega^2 k(k-1)(2k-3)}$), together with the data of Fig. 8.1 as measured in simulations for $\alpha = 2.0$. In both cases the second moment which parametrizes (8.119) and (8.121) was taken from the data: $\mu_2 \approx 1.380$.

$$z(f) = \frac{\log(f) + \frac{1}{2}\log(\mu_2)}{\sqrt{\log(\mu_2)}}. \tag{8.120}$$

$$\text{to } \mathcal{O}(\log^2(\mu_2)) : \ \varrho(f) = \frac{\mathrm{e}^{-\frac{1}{2}z^2(f)}\Big[1+(1/6)\sqrt{\log(\mu_2)}(3z(f) - z^3(f))\Big]}{f\sqrt{2\pi[\log(\mu_2) + \log^2(\mu_2)]}} \tag{8.121}$$

$$z(f) = \frac{\log(f) + (1/2)[\log(\mu_2) + (2/3)\log^2(\mu_2)]}{\sqrt{\log(\mu_2) + \log^2(\mu_2)}}. \tag{8.122}$$

The two theoretical statements (8.119) and (8.121) are indeed found to constitute increasingly accurate predictions for the actual distribution of the relative history frequencies[53] see e.g. Fig. 8.2. We have thus been able to explain the origin and the characteristics of the observed history frequency statistics.

[53] Both results are expansions for small Ω. Should (8.121) be applied to values of Ω that are not small, one has to be careful in dealing with large values of f, where $\varrho(f)$ could become negative (this would have been prevented by the higher orders in Ω, which are absent from the truncated formula). The implication is that the Gaussian integral (8.118) must in practice be complemented by a cut-off $z_c = \mathcal{O}(\Omega^{-1/6})$, i.e. we replace $\int Dz\ [1 + \frac{1}{6}\sqrt{\Omega}(3z - z^3)]F(z) \ \rightarrow \ \int Dz\, g[1 + \frac{1}{6}\sqrt{\Omega}(3z - z^3)]F(z)/\int Dz\, g[1 + \frac{1}{6}\sqrt{\Omega}(3z - z^3)]$, where $g[x] = x\theta[x]$.

8.6.5 The width of $\varrho(f)$

What remains in order to round off the present analysis of the distribution of relative history frequencies is to calculate the width parameter Ω in (8.116) self-consistently from our equations. According to our theory, Ω is given by (8.110), i.e. by

$$\Omega = \sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}}\,\mathrm{Erf}^2\Big[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}}}{\sqrt{2}\sqrt{\zeta^2\kappa^2+(1-\zeta)^2\sigma^2_{\boldsymbol{\lambda}}}}\Big]. \tag{8.123}$$

The quantities $\overline{A}_{\boldsymbol{\lambda}}$ and $\sigma^2_{\boldsymbol{\lambda}} = \overline{A^2}_{\boldsymbol{\lambda}} - \overline{A}_{\boldsymbol{\lambda}}$ describe the statistics of those bids that correspond to times with a prescribed history string $\boldsymbol{\lambda}$. We know from (8.66) that these are Gaussian variables, which implies that $\overline{A}_{\boldsymbol{\lambda}}$ and $\sigma^2_{\boldsymbol{\lambda}}$ are all we need to know. Since we restrict ourselves in this chapter to TTI states without anomalous response, we can write both $\overline{A}_{\boldsymbol{\lambda}}$ and $\overline{A^2}_{\boldsymbol{\lambda}}$ as long-time averages

$$\overline{A}_{\boldsymbol{\lambda}} = \pi_{\boldsymbol{\lambda}}^{-1} \lim_{L\to\infty} L^{-1}\sum_{\ell=1}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)}A(\ell), \tag{8.124}$$

$$\overline{A^2}_{\boldsymbol{\lambda}} = \pi_{\boldsymbol{\lambda}}^{-1} \lim_{L\to\infty} L^{-1}\sum_{\ell=1}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)}A^2(\ell). \tag{8.125}$$

We can work out the average $\overline{A}_{\boldsymbol{\lambda}}$, using (8.66) and TTI, and subsequently define the new time variables $s_i = \ell_i - \ell_{i+1}$ (for $i<r$) and $s_r = \ell_r$ (so that $\ell_j = s_j + s_{j+1} + \cdots + s_r$). This results in

$$\begin{aligned}
\overline{A}_{\boldsymbol{\lambda}} &= \lim_{L\to\infty}\frac{1}{Lp\pi_{\boldsymbol{\lambda}}}\sum_{\ell_0=1}^{L}\sum_{r\geq 0}(-\delta_N)^r\sum_{\ell_1\ldots\ell_r} G(\ell_0-\ell_1),\ldots,G(\ell_{r-1}-\ell_r)\\
&\quad\times\Big[\prod_{i=0}^{r} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_i,A,Z)}\Big]\phi_{\ell_r}\\
&= \frac{1}{p\pi_{\boldsymbol{\lambda}}}\sum_{r\geq 0}(-\delta_N)^r\sum_{s_0,\ldots,s_{r-1}} G(s_0),\ldots,G(s_{r-1})\\
&\qquad \lim_{L\to\infty}\frac{1}{L}\sum_{s_r=0}^{L-\sum_{i<r}s_i}\Big[\prod_{i=0}^{r} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s_i+\cdots+s_r,A,Z)}\Big]\phi_{s_r}.
\end{aligned} \tag{8.126}$$

Given our earlier ansatz of short history correlation times, in the sense of (8.81), and given that $\chi = \sum_{\ell>0} G(\ell) < \infty$ (so that $G(\ell)$ must decay sufficiently fast), we find this expression simplifying to

$$\overline{A}_{\boldsymbol{\lambda}} = \sum_{r\geq 0}(-\chi p\pi_{\boldsymbol{\lambda}})^r \lim_{L\to\infty}\frac{1}{\pi_{\boldsymbol{\lambda}}L}\sum_{s=0}^{L}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s,A,Z)}\phi_s = \frac{\overline{\phi}_{\boldsymbol{\lambda}}}{1+\chi p\pi_{\boldsymbol{\lambda}}}. \tag{8.127}$$

In a similar manner we find

$$
\begin{aligned}
\overline{A^2}_{\boldsymbol{\lambda}} &= \lim_{L\to\infty} \frac{1}{\pi_{\boldsymbol{\lambda}} L} \sum_{\ell_0 \ell_0'=0}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_0,A,Z)} A(\ell_0) A(\ell_0') \delta_{\ell_0 \ell_0'} \\
&= \lim_{L\to\infty} \frac{1}{\pi_{\boldsymbol{\lambda}} L} \sum_{\ell_0 \ell_0'=0}^{L} \sum_{r,r'\geq 0} (-\delta_N)^{r+r'} \sum_{\ell_1,\ldots,\ell_r} G(\ell_0 - \ell_1), \ldots, G(\ell_{r-1} - \ell_r) \\
&\quad \times \sum_{\ell_1',\ldots,\ell_r'} G(\ell_0' - \ell_1'), \ldots, G(\ell_{r'-1}' - \ell_{r'}') \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_0,A,Z)} \\
&\quad \times \left[\prod_{i=1}^{r} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_i,A,Z)} \right] \left[\prod_{i=1}^{r'} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell_i',A,Z)} \right] \delta_{\ell_0 \ell_0'} \phi_{\ell_r} \phi_{\ell_{r'}'} \\
&= \frac{1}{p\pi_{\boldsymbol{\lambda}}} \sum_{r,r'\geq 0} (-\delta_N)^{r+r'} \sum_{s_0,\ldots,s_{r-1}} G(s_0), \ldots, G(s_{r-1}) \sum_{s_0',\ldots,s_{r'-1}'} G(s_0'), \ldots, G(s_{r'-1}') \\
&\quad \times \lim_{L\to\infty} \frac{1}{L} \sum_{s_r=0}^{L-\sum_{i<r} s_i} \sum_{s_{r'}'=0}^{L-\sum_{i<r'} s_i'} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s_0+\cdots+s_r,A,Z)} \\
&\quad \times \left[\prod_{i=1}^{r} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s_i+\cdots+s_r,A,Z)} \right] \left[\prod_{i=1}^{r'} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s_i'+\cdots+s_{r'}',A,Z)} \right] \delta_{\sum_i s_i, \sum_i s_i'} \phi_{s_r} \phi_{s_{r'}'} .
\end{aligned}
\tag{8.128}
$$

Again we use $\sum_\ell G(\ell) < \infty$ to justify that in the summations over s_r and $s_{r'}'$ the upper limit can safely be replaced by L. Thus

$$
\begin{aligned}
\overline{A^2}_{\boldsymbol{\lambda}} &= \frac{1}{p\pi_{\boldsymbol{\lambda}}} \sum_{r,r'\geq 0} (-\delta_N)^{r+r'} \sum_{s_0,\ldots,s_{r-1}} G(s_0), \ldots, G(s_{r-1}) \sum_{s_0',\ldots,s_{r'-1}'} G(s_0'), \ldots, G(s_{r'-1}') \\
&\quad \times \lim_{L\to\infty} \frac{1}{L} \sum_{s_r=0}^{L} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\sum_j s_j,A,Z)} \left[\prod_{i=1}^{r} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\sum_{j\geq i} s_j,A,Z)} \right] \\
&\quad \times \left[\prod_{i=1}^{r'} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\sum_j s_j - \sum_{j<i} s_j',A,Z)} \right] \phi_{s_r} \phi_{\sum_j s_j - \sum_{j<r'} s_j'} .
\end{aligned}
\tag{8.129}
$$

The present calculation is similar to that of the volatility matrix in the online MG,[54] so it is no surprise that here we have to worry about pairwise time coincidences. Each such coincidence effectively removes one constraint of the type $\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ldots,A,Z)}$, since the latter will be met automatically. All the remaining terms will be occurring in extensive summations, so that we may replace each 'unpaired' occurrence of a factor $\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ldots,A,Z)}$,

[54] The quantity $\sigma_{\boldsymbol{\lambda}}^2 = \overline{A^2}_{\boldsymbol{\lambda}} - \overline{A}_{\boldsymbol{\lambda}}^2$ can be regarded as a conditional volatility, where the condition is that in collecting our statistics we are to restrict ourselves to those times where the observed history strings take the value $\boldsymbol{\lambda}$.

except for those with the same argument as one of the Gaussian variables ϕ, by its time average $\pi_{\boldsymbol{\lambda}}$. In practice this implies the replacement

$$\Big[\prod_{i=1}^{r-1} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\sum_{j\geq i}s_j,A,Z)}\Big]\Big[\prod_{i=1}^{r'-1} p\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\sum_j s_j-\sum_{j<i}s'_j,A,Z)}\Big] \rightarrow$$

$$(p\pi_{\boldsymbol{\lambda}})^{r+r'-2}\pi_{\boldsymbol{\lambda}}^{-\sum_{i=1}^{r-1}\sum_{j=1}^{r'-1}\delta_{\sum_{\ell\geq i}s_\ell,\sum_\ell s_\ell-\sum_{\ell<j}s'_\ell}}$$

$$= (p\pi_{\boldsymbol{\lambda}})^{r+r'-2}\prod_{i=1}^{r-1}\prod_{j=1}^{r'-1}\Big[1+(\frac{1}{\pi_{\boldsymbol{\lambda}}}-1)\delta_{\sum_{\ell<i}s_\ell,\sum_{\ell<j}s'_\ell}\Big]$$

$$= (p\pi_{\boldsymbol{\lambda}})^{r+r'-2}\prod_{i=1}^{r-1}\prod_{j=1}^{r'-1}\Big[1+\frac{1-\pi_{\boldsymbol{\lambda}}}{p\pi_{\boldsymbol{\lambda}}}\frac{\tilde{\eta}}{2\delta_N}\delta_{\sum_{\ell<i}s_\ell,\sum_{\ell<j}s'_\ell}\Big] \quad (8.130)$$

and therefore

$$\overline{A^2}_{\boldsymbol{\lambda}} = \sum_{r,r'\geq 0}(-\delta_N)^{r+r'}\sum_{s_0,\ldots,s_{r-1}}G(s_0),\ldots,G(s_{r-1})\sum_{s'_0,\ldots,s'_{r'-1}}G(s'_0),\ldots,G(s'_{r'-1})$$
$$\times (p\pi_{\boldsymbol{\lambda}})^{r+r'}\prod_{i=1}^{r-1}\prod_{j=1}^{r'-1}\Big[1+\frac{1-\pi_{\boldsymbol{\lambda}}}{p\pi_{\boldsymbol{\lambda}}}\frac{\tilde{\eta}}{2\delta_N}\delta_{\sum_{\ell<i}s_\ell,\sum_{\ell<j}s'_\ell}\Big]$$
$$\times \sum_k \delta_{k,\sum_{j<r}s_j-\sum_{j<r'}s'_j}\lim_{L\to\infty}\frac{1}{\pi_{\boldsymbol{\lambda}}^2 L}\sum_{s=0}^{L}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s,A,Z)}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s+k,A,Z)}\phi_s\phi_{s+k}. \quad (8.131)$$

As in the calculation of the volatility for the on-line MG, lacking as yet a method to deal with all the complicated terms generated by the factor proportional to the learning rate $\tilde{\eta}$, we have to restrict ourselves in practice to approximations. As before we first remove the most tricky terms by putting $\tilde{\eta}\to 0$. This gives

$$\overline{A^2}_{\boldsymbol{\lambda}} = \sum_{r,r'\geq 0}(-\delta_N p\pi_{\boldsymbol{\lambda}})^{r+r'}\sum_{s_0,\ldots,s_{r-1}}G(s_0),\ldots,G(s_{r-1})\sum_{s'_0,\ldots,s'_{r'-1}}G(s'_0),\ldots,G(s'_{r'-1})$$
$$\times \sum_k \delta_{k,\sum_{j<r}s_j-\sum_{j<r'}s'_j}\lim_{L\to\infty}\frac{1}{\pi_{\boldsymbol{\lambda}}^2 L}\sum_{s=0}^{L}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s,A,Z)}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s+k,A,Z)}\phi_s\phi_{s+k}. \quad (8.132)$$

We subsequently assume that the limit $L\to\infty$ in the last line converts the associated sample average into a full average over the statistics of the Gaussian fields ϕ_ℓ as given by (8.52), i.e.

$$\lim_{L\to\infty}\frac{1}{\pi_{\boldsymbol{\lambda}}^2 L}\sum_{s=0}^{L}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s,A,Z)}\delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s+k,A,Z)}\phi_s\phi_{s+k}$$

$$\to \lim_{L\to\infty} \frac{1}{\pi_{\boldsymbol{\lambda}}^2 L} \sum_{s=0}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s,A,Z)} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(s+k,A,Z)} \langle \phi_s \phi_{s+k} \rangle_{\phi|A,Z} = \frac{1}{2}[1 + C(k)].$$

If we also separate the correlation function into a persistent and a non-persistent term, via $C(k) = c + \tilde{C}(k)$, and return to the earlier notation with time differences inside the kernels G, the result is the history-conditioned equivalent of our earlier fake history volatility approximation stage (5.90):

$$\begin{aligned}
\overline{A^2}_{\boldsymbol{\lambda}} &= \sum_{r,r'\geq 0} (-\delta_N p \pi_{\boldsymbol{\lambda}})^{r+r'} \sum_{s_0,\ldots,s_{r-1}} G(s_0),\ldots,G(s_{r-1}) \sum_{s'_0,\ldots,s'_{r'-1}} G(s'_0),\ldots,G(s'_{r'-1}) \\
&\quad \times \frac{1}{2}\Big[1 + c + \tilde{C}\Big(\sum_{j<r} s_j - \sum_{j<r'} s'_j\Big)\Big] \\
&= \frac{1+c}{2(1+\chi p \pi_{\boldsymbol{\lambda}})^2} + \frac{1}{2}\int \mathrm{d}s\mathrm{d}s' \, (\mathbf{I} + p\pi_{\boldsymbol{\lambda}} G)^{-1} \tilde{C}(s-s') (\mathbf{I} + p\pi_{\boldsymbol{\lambda}} G)^{-1},
\end{aligned} \tag{8.133}$$

where, as always, $\mathbf{I}(x,y) = \delta(x-y)$. In order to get to the present stage we have averaged the ϕ-dependent terms inside $\overline{A^2}_{\boldsymbol{\lambda}}$ over the Gaussian measure $\langle \ldots \rangle_{\phi|A,Z}$. Consistency demands that in working out $\sigma_{\boldsymbol{\lambda}}^2 = \overline{A^2}_{\boldsymbol{\lambda}} - \overline{A}_{\boldsymbol{\lambda}}^2$ we do the same with the term $\overline{A}_{\boldsymbol{\lambda}}^2$, where $\overline{A}_{\boldsymbol{\lambda}}$ is given by (8.127), so that our present approximation for the history-conditioned volatility becomes

$$\begin{aligned}
\boldsymbol{\sigma}_{\boldsymbol{\lambda}}^2 &= \overline{A^2}_{\boldsymbol{\lambda}} - \frac{\langle \overline{\phi}_{\boldsymbol{\lambda}}^2 \rangle}{(1+\chi p \pi_{\boldsymbol{\lambda}})^2} \\
&= \overline{A^2}_{\boldsymbol{\lambda}} - \lim_{L\to\infty} \frac{1}{(L\pi_{\boldsymbol{\lambda}})^2} \sum_{\ell\ell'=1}^{L} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell,A,Z)} \delta_{\boldsymbol{\lambda},\boldsymbol{\lambda}(\ell',A,Z)} \frac{1 + C(\ell-\ell')}{2(1+\chi p \pi_{\boldsymbol{\lambda}})^2} \\
&= \overline{A^2}_{\boldsymbol{\lambda}} - \frac{1+c}{2(1+\chi p \pi_{\boldsymbol{\lambda}})^2} \\
&= \frac{1}{2}\int \mathrm{d}s\mathrm{d}s' \, (\mathbf{I} + p\pi_{\boldsymbol{\lambda}} G)^{-1}(s) \tilde{C}(s-s') (\mathbf{I} + p\pi_{\boldsymbol{\lambda}} G)^{-1}(s').
\end{aligned} \tag{8.134}$$

Our final approximation step is identical to that which took us earlier from (5.90) to (5.91). We assume that the non-persistent correlations $\tilde{C}(t)$ decay very fast, away from the value $\tilde{C}(0) = 1 - c$, so that in the expansion of (8.134) in powers of G we retain only the zero-th term:

$$\boldsymbol{\sigma}_{\boldsymbol{\lambda}}^2 = \frac{1}{2}(1-c). \tag{8.135}$$

We can now return to our expression (8.123) for Ω, and insert our two approximate results (8.127) and (8.135), giving:

$$\begin{aligned}\Omega &= \lim_{p\to\infty}\sum_{\boldsymbol{\lambda}}\pi_{\boldsymbol{\lambda}}\,\mathrm{Erf}^2\left[\frac{(1-\zeta)\overline{\phi}_{\boldsymbol{\lambda}}}{\sqrt{2}(1+\chi p\pi_{\boldsymbol{\lambda}})\sqrt{\zeta^2\kappa^2+(1/2)(1-\zeta)^2(1-c)}}\right]\\ &= \int \mathrm{d}f\mathrm{d}\phi\;\varrho(f,\phi)f\;\mathrm{Erf}^2\left[\frac{(1-\zeta)\phi}{\sqrt{2}(1+\chi f)\sqrt{\zeta^2\kappa^2+(1/2)(1-\zeta)^2(1-c)}}\right]\end{aligned} \tag{8.136}$$

with

$$\varrho(f,\phi)=\lim_{p\to\infty}\frac{1}{p}\sum_{\boldsymbol{\lambda}}\delta[f-p\pi_{\boldsymbol{\lambda}}]\delta[\phi-\overline{\phi}_{\boldsymbol{\lambda}}]. \tag{8.137}$$

We know the $\overline{\phi}_{\boldsymbol{\lambda}}$ to be Gaussian variables, with $\langle\overline{\phi}_{\boldsymbol{\lambda}}\rangle=0$ and $\langle\overline{\phi}_{\boldsymbol{\lambda}}^2\rangle=\frac{1}{2}(1+c)$ (see the above derivation of $\sigma_{\boldsymbol{\lambda}}^2$ where this was shown and used). Hence, upon making our final simplifying assumption that in the relevant orders of our present calculation the possible correlations between the history frequencies $\pi_{\boldsymbol{\lambda}}$ and the Gaussian fields $\overline{\phi}_{\boldsymbol{\lambda}}$ are irrelevant, we obtain

$$\varrho(f,\phi)=\varrho(f)\;\frac{\mathrm{e}^{-\phi^2/(1+c)}}{\sqrt{\pi(1+c)}}, \tag{8.138}$$

and hence (8.136) simplifies to

$$\Omega=\int_0^\infty \mathrm{d}f\;\varrho(f)f\int Dx\;\mathrm{Erf}^2\left[\frac{x(1-\zeta)\sqrt{1+c}}{2(1+\chi f)\sqrt{\zeta^2\kappa^2+(1/2)(1-\zeta)^2(1-c)}}\right]. \tag{8.139}$$

The integral $\int Dx\;\mathrm{Erf}^2(Ax)$ is calculated in Appendix B, see (B.17). It is found to be equal to $\frac{4}{\pi}\arctan[\sqrt{1+4A^2}]-1$, so that, in combination with the general identity $\int \mathrm{d}f\;\varrho(f)f=1$, our approximate expression for the width parameter Ω becomes

$$\Omega=\frac{4}{\pi}\int_0^\infty \mathrm{d}f\;\varrho(f)f\;\arctan\left[1+\frac{(1-\zeta)^2(1+c)}{(1+\chi f)^2[\zeta^2\kappa^2+(1/2)(1-\zeta)^2(1-c)]}\right]^{(1/2)}-1. \tag{8.140}$$

In the limit of strictly fake history we recover from (8.140) the value $\lim_{\zeta\to 1}\Omega = (4/\pi)\arctan[1]-1=0$, as it should. For MGs with strictly true market history, on the other hand, expression (8.140) simplifies to

$$\lim_{\zeta\to 0}\Omega = \frac{4}{\pi}\int_0^\infty \mathrm{d}f\ \varrho(f)f\ \arctan\left[1+\frac{2(1+c)}{(1+\chi f)^2(1-c)}\right]^{1/2} - 1. \quad (8.141)$$

In accordance with our observations in simulations we also observe that, as the system approaches the phase transition when α is lowered from within the ergodic regime, the increase of the susceptibility χ automatically reduces the width parameter Ω, until it vanishes completely at the critical point.

8.7 Closed theory in the ergodic regime

We have now obtained a closed theory for the TTI states, albeit in approximation. It consists of the equations (8.87)–(8.90) for the persistent order parameters, combined with expressions (8.119) and (8.121) for the shape and (8.117) and (8.140) for the width of the relative history frequency distribution $\varrho(f)$. This theory predicts correctly that the phase transition point $\alpha_c(T)$ of the MG with history is identical to that of the model with fake memory, that at the transition point the relative history frequency distribution reduces to $\varrho(f)=\delta[f-1]$ (also at that point the order parameters all become independent of whether we have true or fake history), and it predicts more or less correctly the shape of the relative history frequency distribution. In the limit $\alpha\to\infty$ the theory also reproduces the correct order parameter values $\chi=\phi=c=0$, for any value of ζ. For $\zeta=0$ (strictly true memory) it predicts $\lim_{\alpha\to\infty}\Omega=\frac{1}{3}$ and hence $\lim_{\alpha\to\infty}\mu_2\approx 1.37$.

Let us finally work out our closed equations, and try to reduce them to a more compact form, for the simplest non-trivial case of the MG with strictly true market history (i.e. $\zeta=0$) and without decision noise (i.e. $\sigma[\infty]=1$). Here we have

$$u=\frac{\sqrt{\alpha}\chi_R}{S_0\sqrt{2}},\qquad \chi=\frac{1-\phi}{\alpha\chi_R},\qquad \phi=1-\mathrm{Erf}[u], \quad (8.142)$$

$$c=1-\mathrm{Erf}[u]+\frac{1}{2u^2}\mathrm{Erf}[u]-\frac{1}{u\sqrt{\pi}}\mathrm{e}^{-u^2}, \quad (8.143)$$

$$\chi_R=\int Dz\left[1+\frac{1}{6}\sqrt{\Omega}(3z-z^3)\right]\left[\mathrm{e}^{-z\sqrt{\Omega+\frac{5}{6}\Omega^2}+\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}+\chi\right]^{-1}, \quad (8.144)$$

$$S_0^2=(1+c)\int Dz\left[1+\frac{1}{6}\sqrt{\Omega}(3z-z^3)\right]\left[\mathrm{e}^{-z\sqrt{\Omega+\frac{5}{6}\Omega^2}+\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}+\chi\right]^{-2}, \quad (8.145)$$

$$\Omega=\int Dz\left[1+\frac{1}{6}\sqrt{\Omega}(3z-z^3)\right]\mathrm{e}^{z\sqrt{\Omega+\frac{5}{6}\Omega^2}-\frac{1}{2}(\Omega+\frac{1}{2}\Omega^2)}$$

$$\times \left\{ \frac{4}{\pi} \arctan \left[1 + \frac{2(1+c)}{(1-c)\left[1+\chi e^{z\sqrt{\Omega+(5/6)\Omega^2}-(1/2)(\Omega+\frac{1}{2}\Omega^2)}\right]^2} \right]^{1/2} - 1 \right\}. \tag{8.146}$$

Upon using equation (8.143) to write c as a function of u, i.e. $c = c(u)$ with $c(u)$ denoting the right-hand side of (8.143), and upon eliminating the quantities ϕ and S_0, we find ourselves with a closed set of equations for the trio $\{u, \chi, \Omega\}$:

$$u = \frac{\text{Erf}[u]}{\chi\sqrt{2\alpha(1+c)}} \left\{ \int Dz \frac{1+(1/6)\sqrt{\Omega}(3z-z^3)}{\left[e^{-z\sqrt{\Omega+(5/6)\Omega^2}+(1/2)(\Omega+\frac{1}{2}\Omega^2)}+\chi\right]^2} \right\}^{-1/2}, \tag{8.147}$$

$$\chi = \frac{\text{Erf}[u]}{\alpha} \left\{ \int Dz \frac{1+(1/6)\sqrt{\Omega}(3z-z^3)}{e^{-z\sqrt{\Omega+(5/6)\Omega^2}+(1/2)(\Omega+(1/2)\Omega^2)}+\chi} \right\}^{-1}, \tag{8.148}$$

$$\Omega = \int Dz \left[1 + \frac{1}{6}\sqrt{\Omega}(3z-z^3) \right] e^{z\sqrt{\Omega+(5/6)\Omega^2}-(1/2)(\Omega+(1/2)\Omega^2)}$$
$$\times \left\{ \frac{4}{\pi} \arctan \left[1 + \frac{2[1+c(u)]}{[1-c(u)]\left[1+\chi e^{z\sqrt{\Omega+(5/6)\Omega^2}-(1/2)(\Omega+(1/2)\Omega^2)}\right]^2} \right]^{1/2} - 1 \right\}. \tag{8.149}$$

Solving these three coupled equations numerically, followed by comparison with simulation data, shows a rather surprising level of agreement, given the tedious expansions and assumptions that have been required in order to get to (8.147)–(8.149). Figure 8.3, for example, shows the performance of the theory in describing the on-line MG with strictly true market history (i.e. $\zeta = 0$), together with the earlier data for the on-line fake history MG (i.e. $\zeta = 1$), for comparison.[55] Calculation of the first two non-trivial moments μ_k of the distribution of relative history frequencies, see Fig. 8.4, shows that for small values of the width of $\varrho(f)$ (i.e. for μ_2 close to one, which is true close to and below the critical point) the predictions of the theory are excellent, but that the theory cannot handle large values of μ_2. This is obvious and inevitable, given that the theory was essentially developed as an expansion for small values of the width $\mu_2 - 1$ of the distribution $\varrho(f)$. Taking the various expansions to a higher order in Ω should lead to systematic improvement, but it will involve non-trivial calculations.

[55] Note that below the critical point, where $\chi = \infty$ throughout, equation (8.140) predicts that $\Omega = 0$. This implies that $\varrho(f) = \delta[f-1]$ for $\alpha < \alpha_c(T)$, and that below the critical point the differences between true and fake history (if any) must therefore be confined to dynamical phenomena or to states without TTI. This confirms our earlier observations in numerical simulations, where we found that the persistent order parameters in MGs with and MGs without history were identical in the low α regime.

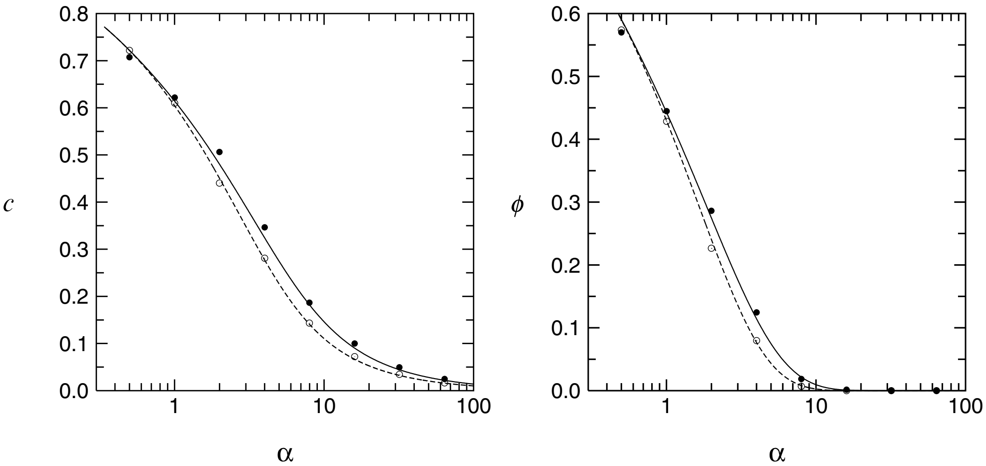


Fig. 8.3 Left: the predicted persistent correlations c together with simulation data in the non-ergodic regime, for the on-line MG with strictly true history (i.e. $\zeta = 0$; the solid line gives the theoretical prediction, full circles the experimental data) and for the on-line MG with strictly fake memory (i.e. $\zeta = 1$; the dashed line gives the theoretical prediction, open circles the experimental data). In both cases decision noise was absent. Right: the corresponding predicted fraction ϕ of frozen agents, under the same experimental conditions and with the same meaning of lines and markers.

8.8 An assessment

It will be clear that in the present chapter we have had to make a significant number of ansätze and assumptions, especially when compared with previous chapters. This is simply a direct consequence of the serious complications generated by reintroducing true memory into our microscopic equations; here no simple theory will lead to correct predictions derived from 'first principles'. Given the rather tedious calculations required, one might also wonder whether the relatively minor differences between the solid curves (true history) and the dashed curves (fake history) in Fig. 8.3 were actually worth the time and energy invested in deriving equations (8.147)–(8.149). To this there is a simple answer: although here, in the simple standard MG, the differences between supplying real or fake market data to the agents are minor at the level of macroscopic observables, it is not difficult to devise MG versions where these differences become prominent and important. We now at least understand the mechanism responsible for these differences, and have developed the mathematical tools to predict their macroscopic consequences, also in alternative MG versions.

In the preamble of this chapter we posed two questions, the first mainly of relevance to those interested in statistical mechanical model solving as such, and the second from the perspective of those interested mainly in the phenomenology of markets: whether

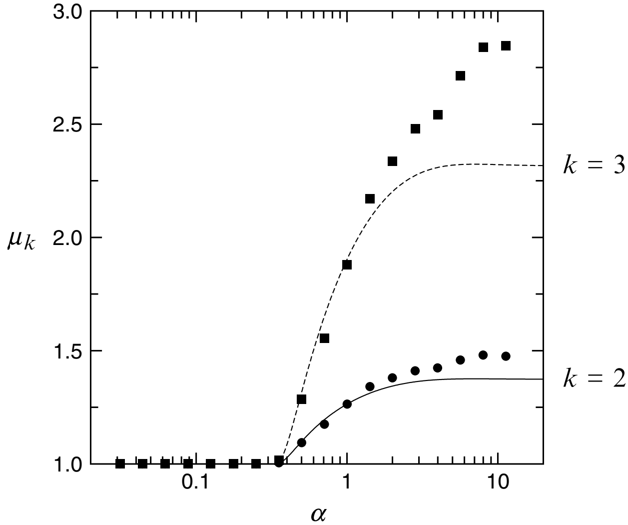


Fig. 8.4 The moments $\mu_2 = \int \mathrm{d}f \ \varrho(f) f^2$ and $\mu_3 = \int \mathrm{d}f \ \varrho(f) f^3$ of the distribution of relative history frequencies for the MG with strictly true history and absent decision noise (i.e. $\zeta = T = 0$), as predicted by the theory (solid and dashed lines), compared with the moments as measured in numerical simulations (markers, with circles indicating μ_2 and squares indicating μ_3). Note that $\mu_0 = \mu_1 = 1$ (by definition).

and how the generating functional formalism can be adapted in order to deal with history in the MG, and what is the explanation for finding only small differences between MGs with true versus fake market memory.

The answer to the first question is clear. The generating functional method again applies, but here it leads to a theory in terms of two non-Markovian effective processes (rather than one, as in models with fake histories), the first describing the evolution of an effective agent and the second describing that of an effective overall bid. Finding solutions of these exact order parameter equations is harder than in models with fake histories, and evolves around the extraction of history recurrence statistics from the effective bid process.

The answer to the second question would seem to be as follows. The equations of the MG have been constructed such that the agents continually try to remove overall bid predictability from the market. This more or less implies making the overall bid evolution as random as possible. They will succeed in doing so as long as the spectrum of strategies in the system is sufficiently broad to allow for any emerging pattern to be picked up by the agents, exploited, and thereby destroyed (which ought to happen when N is large for a given value of M, i.e. for $\alpha = 2^M/N$ small). We see in the present theory that the agents always succeed in reducing the possible impact of having real histories to at most a non-uniform history frequency distribution, for any α, where the remaining differences in history frequencies (if present) reflect the possibility that some bid patterns may be more difficult to remove than others as a consequence of not having enough agents in the market. When the number of agents is sufficiently large to also remove the 'difficult' patterns (i.e. for $\alpha < \alpha_{\rm c}$) the agents are able to remove *all* predictability and we return to a situation equivalent to having fake memory. Therefore, the observation that the differences between the behaviour of MGs with real versus fake market histories are small simply tells us that the agents are surprisingly efficient at doing exactly what they were designed to do, and the absence of such differences for small α confirms that having more agents indeed enables the market to draw upon a larger pool of strategies, which makes it easier to destroy predictable bid patterns. We would expect differences to emerge in models where the agents are being frustrated in their efforts to reduce predictability.

9. Variations and generalizations

The initial motivation to introduce and study MG was largely to increase our understanding of the collective effects of inductive decision making by large numbers of agents in (financial) markets. It is therefore no surprise that the model's conception subsequently generated a large body of generalizations, variations, and spin-off models, which all tried to introduce further details into the basic MG in order to bring the model closer to economic reality. It is neither possible nor helpful to give an exhaustive overview of all such variations and generalizations which have been published. Instead, here we will limit ourselves to a selection that has been made largely on the basis of whether the proposed variation not only makes sense but also allows for the resulting model to be solved and understood mathematically with the techniques presently available.

The remaining 'solvable' models, which all aim at bringing diversity into the population of agents, can more or less be divided into two distinct groups. The first group consist of generalizations that at the mathematical level are relatively easy to incorporate, as they affect only relatively simple parameters of the effective single agent process. Examples are allowing the agents to have different levels of decision noise or different learning rates, or by allowing some agents to act as 'trend followers' rather than individuals seeking to be in the minority. In the second group of generalizations the proposed modifications of the standard MG affect the strategy statistics, and have a potentially bigger impact on the structure of the theory. One example is a category of models where agents are divided into types: producers (with only one strategy each), speculators (which are similar to the agents in the standard MG, but are now also equipped with the option to refrain from trading, which can itself be implemented as an additional 'default' strategy), and sometimes also a third category of 'noisy' agents who simply act randomly.

Given that the above new ingredients will inevitably complicate our mathematical analysis, given the technical difficulties associated with having true market memory already in the basic MG, and given the relatively modest differences that have so far been observed in MGs with true market histories compared with the fake history versions,[56] all of the generalized but still solvable MG models have been of the fake history type. To keep matters as simple as possible, and hence focus on the generalizations themselves, we will restrict our analysis solely to microscopic equations of the batch type, i.e. with discrete time and with averages over all possible fake history strings at each iteration step.

9.1 Which generalization should one be looking for?

9.1.1 The stylized facts of financial time series

In order to steer us in the right direction it will be useful to first reflect upon the main features that more realistic market models should exhibit. If we allow ourselves to be guided by the phenomenology of financial markets in particular, it strikes us immediately that the time series of the overall bid in the MG versions discussed so far (making for the moment the reasonable assumption that the overall bid $A(\ell)$ in the MG must be related in some way to an asset price) as much too regular. Although the bid $A(\ell)$ evolves at first sight randomly, this randomness can be described by smooth distributions of limited width, which do not allow for 'rare events'. There are no fluctuations with envelopes that vary significantly over time, nor avalanches or crashes; processes in the real (financial) world are simply much more messy and irregular than those in the conventional MG versions. More specifically, in financial time series $\{x(t)\}$ one observes certain types of consistent qualitative features and statistical characteristics, called 'stylized facts'. These accepted empirical truths have acquired the status of benchmarks, to be met by any theory claiming to explain aspects of financial time series. The two main ones are

Volatility clustering

The oscillations in the observed time series of quantities such as prices or shares are not having uniform envelopes, but show pseudo-tranquil periods alternating with 'bursts' and even catastrophic events such as crashes. This is usually reflected in algebraic decay as a function of τ of the (positive) auto-correlation function of the returns:[57]

[56] This logic is clearly somewhat dangerous, since the effects of true histories could become very different if we move away from the recipes of the standard MG. Two examples where having true history is claimed to induce significant differences are MGs with self-impact correction (as in Chapter 8) and MGs where different agents are allowed different values of M (or, equivalently, where the strategies assigned to the agents are chosen such that they are not all equally able to distinguish between different history strings).

[57] The price return $r(\ell)$ at stage ℓ of a price time series $\{x(\ell)\}$ is in theoretical studies usually defined as $r(\ell) = \log[x(\ell+1)/x(\ell)]$, which is the definition that we shall adopt here. One sometimes also encounters slightly alternative definitions, such as the more down-to-earth $r(\ell) = [x(\ell+1) - x(\ell)]/x(\ell)$ or even simply $r(\ell) = x(\ell+1) - x(\ell)$.

$$C_r(\tau) = \lim_{t\to\infty} \frac{\sum_{\ell=1}^{t-\tau}\langle r(\ell+\tau)r(\ell)\rangle}{\sum_{\ell=1}^{t-\tau}\langle r^2(\ell)\rangle}. \tag{9.1}$$

Leptokurtosis ('fat tails')

If one measures over some large period L of trading activity quantities such as the distribution $P(r) = L^{-1}\sum_{\ell\leq L}\delta[r - r(\ell)]$ of the price returns or the distribution $P(V) = L^{-1}\sum_{\ell\leq L}\delta[V - V(\ell)]$ of the total volumes of trade in the market, one typically finds distributions that are far from Gaussian.[58] They decay to zero monotonically as one moves away from the maximum, but much too slowly, according to power laws.

There are several more such 'stylized facts', such as timescale invariance of the distribution of returns, absence of short-time auto-correlations in the prices themselves (in contrast to the price volatility), leverage effects (negative correlations between volatilities and returns), etc.

9.1.2 Economic interpretation of the MG

We must now translate these empirical statements into MG language, upon assuming that the MG indeed models (in an as yet primitive way) the trading of assets in a market. At the present stage, however, the MG has not yet been equipped with explicit economic variables. In particular, we will need to define the concepts of *asset price* and *trading volume* in terms of the original microscopic variables of the MG. This can obviously be done in different ways, but it would seem that, as a first basic level of sophistication, consensus has more or less been reached on the following definitions:

Definition of asset price in the MG

Given our interpretation of a bid $b_i(\ell) < 0$ in the MG as sell and $b_i(\ell) > 0$ as buy, we have for $A(\ell) < 0$ an excess of sellers (which must drive the price x down) and for $A(\ell) > 0$ an excess of buyers (which must drive the price up). Thus the price change $x(\ell+1) - x(\ell)$ should be a monotonically increasing function of $A(\ell)$. A simple form agreed upon[59] is $\log[x(\ell+1)] = \log[x(\ell)] + A(\ell)/\Lambda$. Equivalently:

$$x(\ell+1) = x(\ell)\,\mathrm{e}^{A(\ell)/\Lambda}. \tag{9.2}$$

Here $\Lambda > 0$ is some asset-specific constant that controls the sensitivity of the asset price to demand/supply imbalance. Formula (9.2) not only gives us a price definition

[58] The term *leptokurtosis* refers to the so-called kurtosis κ of a stochastic variable x, which is defined as $\kappa = \langle(x-\langle x\rangle)^4\rangle/\langle(x-\langle x\rangle)^2\rangle^2 - 3$. For all Gaussian distributions one would have $\kappa = 0$. A positive value for κ indicates that the decay of the distribution for x as $x \to \pm\infty$ is slower than that of a Gaussian distributed variable.

[59] The arguments leading to these definitions are of an economic nature, and can be found in the more specialized literature to which pointers will be given later. Clearly, also alternative choices have been proposed, such as the even simpler relation $x(\ell+1) = x(\ell) + A(\ell)/\Lambda$.

for the MG, but it also has the convenient consequence that (apart from a constant) the returns $r(\ell)$ are simply identical to the overall bids $A(\ell)$ in the MG.

Definition of trading volume in the MG

Here one simply adds up the total trading activity at a given iteration of the game, irrespective of its nature (whether buy or sell), i.e.

$$V(\ell) = \frac{1}{N}\sum_i |b_i(\ell)|. \tag{9.3}$$

The rescaling with N, which is of no consequence for the purpose of investigating presence or absence of stylized facts, has been introduced in order to have $V(\ell) = \mathcal{O}(N^0)$, in view of the limit $N \to \infty$ in our analyses.

It also follows from the interpretation (9.2), given self-averaging of the volatility matrix Ξ in the MG, that in TTI situations the returns' auto-correlation function (9.1) is equal to $C_r(\tau) = \overline{\Xi}(\tau)/\overline{\Xi}(0)$. We are now in a position to translate the main stylized facts of financial time series into desired properties of our MG models. This results in the following shopping list:

(i) *Volatility clustering—bursting*
The time series of the overall bid $A(\ell)$ in the MG must exhibit pseudo-tranquil periods alternating with 'bursts', and possibly crashes.

(ii) *Volatility clustering—algebraic decay of returns' auto-correlations*
The TTI disorder-averaged volatility matrix $\overline{\Xi}(\tau)$ must decay to zero algebraically for $\tau \to \infty$.

(iii) *Fat tails in returns distribution*
The stationary overall bid distribution $P(A)$ must decay algebraically for $A \to \pm\infty$.

(iv) *Fat tails in volume distribution*
The stationary distribution $P(V)$ of the volumes $V(\ell) = N^{-1}\sum_i |b_i(\ell)|$ must decay algebraically for $V \to \infty$.

In the standard batch MG[60] the bid distribution $P(A)$ always evolves towards a Gaussian shape, on finite timescales. In the on-line MG, in contrast, it appears that one does indeed have non-Gaussian overall bid statistics in the stationary state, see Fig. 6.3. Although the behaviour of the outliers of $P(A)$ in the limit $N \to \infty$ is still unresolved, the central part of the distribution $P(A)$ for on-line MG versions with and

[60] One can easily convince oneself that the calculation in Chapter 6 of the overall bid distribution $P(A)$ of the batch MG applies equally to the batch version with self-impact correction of Chapter 7. Hence also in the latter model one will always have Gaussian bid distributions.

without true market histories did suggest non-Gaussian tails. Whether these are indeed algebraic (i.e. $P(A) \sim |A|^{-\gamma}$) or perhaps exponential (i.e. $P(A) \sim \mathrm{e}^{-\mu|A|}$, which at this stage appears more likely) would seem difficult to tell with certainty on the basis of the data presently available (one obviously needs very large system sizes and abundant data to probe the tails of $P(A)$). However, true bursts in the evolution of the bids $A(\ell)$ have so far been conspicuously absent in the standard MG versions. Furthermore, except for the spherical MG (which in the stationary state has a volatility matrix of the form $\overline{\Xi}(t) = a + b(-1)^t$, so that this version is ruled out on economic realism), the MG versions studied so far have always had binary bids $b_i(\ell) \in \{-1, 1\}$, so there definition (9.3) would have reduced simply to $V(\ell) = 1$, and the volume distribution is just a δ-peak, $P(V) = \delta(V - 1)$. One could argue that volatility clustering and fat tails for the bid distribution might be more likely to show up at the phase transition transition point of the (on-line) MG. This, however, is not found to be the case; see Fig. 9.1.

We must conclude that the conventional MG versions discussed so far do not yet exhibit the main stylized facts of financial markets, except perhaps (iii), 'fat tails' in the bid distribution $P(A)$ for on-line MG versions, which are indeed observed in the non-ergodic low α regime. The standard MG versions are simply too well-behaved. New ingredients are apparently required if we aim for economic realism. The most popular types of revision of the standard MG have been based on introducing forms of diversity into the community of agents.

9.2 MGs with population diversity and trend followers

9.2.1 Definitions

Let us first return to equations (4.2) and (4.3) for the simple batch MG with 'fake' market memory and two strategies per agent. In this standard model we now implement two generalizations, in order to account for the fact that not all agents operate according to identical principles (in addition to the differences in their private strategies, in terms of which agents have always been distinct, even in the standard MG). First, rather than having one uniform noise strength parameter T in the decision noise definitions (2.12) and (2.13), we will here allow each agent to be characterized by an individual noise parameter T_i, so that we now have to add the noise strength as a condition in our definitions:

$$\text{additive}: \qquad \sigma[q, z|T] = \mathrm{sgn}[q + Tz], \tag{9.4}$$

$$\text{multiplicative}: \qquad \sigma[q, z|T] = \mathrm{sgn}[q]\ \mathrm{sgn}[1 + Tz]. \tag{9.5}$$

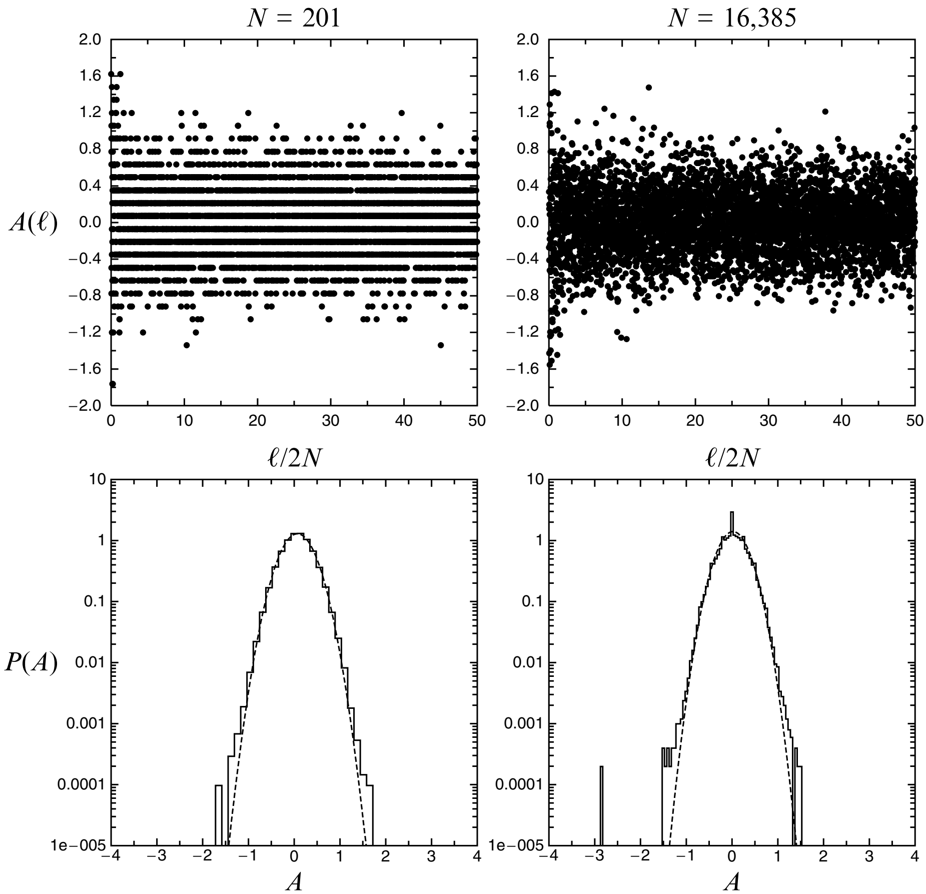


Fig. 9.1 Top: numerical simulation measurements of the re-scaled total bids $A(\ell)$ in the ordinary on-line MG with fake histories and $S = 2$, but operating at the critical point $\alpha = 0.3374$. Bottom: the corresponding distributions $P(A)$ of these observed values (represented as histograms, and plotted logarithmically), together with Gaussian distributions (dashed) of width and average identical to those of the observed $P(A)$. Clearly, neither small ($N = 201$) nor large systems ($N = 16,385$) exhibit true volatility clustering or fat tails in the bid (i.e. price) distribution.

Thus agents differ in the extent to which they allow their decisions to be ruled by their instantaneous strategy valuation differences. Second, we no longer take all agents to be driven by the desire to be in the minority, but we will allow also for a fraction of 'trend followers'. The latter agents value their individual strategies by rewarding *majority* decisions, rather than minority decisions. If we introduce binary labels $\varepsilon_i \in \{-1, 1\}$, with $\varepsilon_i = -1$ for the traditional 'minority seeking' agents and with $\varepsilon_i = 1$ for the new 'majority seeking' agents, we find our combined generalizations leading to the following microscopic MG process

$$q_i(t+1) = q_i(t) + \frac{2\varepsilon_i}{\sqrt{N}} \sum_{\mu=1}^{p} \xi_i^\mu A^\mu[\mathbf{q}(t), \mathbf{z}(t)], \tag{9.6}$$

$$A^\mu[\mathbf{q}, \mathbf{z}] = \Omega_\mu + \frac{1}{\sqrt{N}} \sum_i \sigma[q_i, z_i|T_i]\xi_i^\mu. \tag{9.7}$$

Apart from the replacement $\sigma[q_i, z_i] \to \sigma[q_i, z_i|T_i]$, nothing changes in the original definitions of the volatility matrix, nor in that of the strategy statistics. The disorder-averaged correlation and response functions simply generalize to

$$C_{tt'} = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \sigma[q_i(t), z_i(t)|T_i]\ \sigma[q_i(t'), z_i(t')|T_i]\rangle}, \tag{9.8}$$

$$G_{tt'} = \lim_{N\to\infty} \frac{1}{N} \sum_i \frac{\partial}{\partial\theta_i(t')} \overline{\langle \sigma[q_i(t), z_i(t)|T_i]\rangle}. \tag{9.9}$$

In the present calculation we will also encounter a slightly modified response function $\hat{G}$, which is defined as

$$\hat{G}_{tt'} = -\lim_{N\to\infty} \frac{1}{N} \sum_i \varepsilon_i \frac{\partial}{\partial\theta_i(t')} \overline{\langle \sigma[q_i(t), z_i(t)|T_i]\rangle}, \tag{9.10}$$

and which is seen to reduce to (9.9) in the case where the fraction of trend followers vanishes. Our present non-uniform agent population can be described by the following joint distribution, with $\varepsilon \in \{-1, 1\}$ and $T \in [0, \infty\rangle$:

$$W(\varepsilon, T) = \lim_{N\to\infty} \frac{1}{N} \sum_i \delta_{\varepsilon,\varepsilon_i} \delta[T - T_i]. \tag{9.11}$$

It will turn out that knowledge of $W(\varepsilon, T)$ suffices to adapt our previous solution of the batch MG to the present model with agent diversity and trend followers. We would return to the simpler model of Chapter 4 upon choosing W to be of the form $W(\varepsilon, T) = \delta_{\varepsilon,-1}\delta[T - T^\star]$.

Whereas the introduction of agent-specific noise levels should be expected to lead at most to quantitative changes in the MG's macroscopic behaviour (in the limit $N \to \infty$), the introduction of trend followers could have more drastic effects since it introduces positive feedback into the microscopic laws (similar to the introduction of market self-impact correction in Chapter 7).

9.2.2 The disorder-averaged generating functional

In order for generalized correlation and response functions (9.8) and (9.9) to be generated, we only have to make the appropriate substitution $\sigma[q_i, z_i] \to \sigma[q_i, z_i|T_i]$ in the functional (4.9) as defined for the standard batch MG

$$Z[\boldsymbol{\psi}] = \langle \mathrm{e}^{\mathrm{i}\sum_{t\geq 0}\sum_i \psi_i(t)\sigma[q_i(t),z_i(t)|T_i]}\rangle. \tag{9.12}$$

Especially upon enlisting the shorthand $s_i(t) = \sigma[q_i(t), z_i(t)|T_i]$, we find that very little changes in most of the derivations of Chapter 4, beyond simply inserting the factors $-\varepsilon_i$ in the right places. We find (4.16) and (4.17) being replaced by

$$\begin{aligned} Z[\boldsymbol{\psi}] &= \Big\langle \int p_0(\mathbf{q}(0))\Big[\prod_{it}\frac{\mathrm{d}q_i(t)\mathrm{d}\hat{q}_i(t)}{2\pi}\mathrm{e}^{-\mathrm{i}\varepsilon_i\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)]+\mathrm{i}\psi_i(t)s_i(t)}\Big] \\ &\quad\times\prod_{ij\mu}\mathrm{e}^{\frac{2i}{N}\xi_i^\mu\sum_t\hat{q}_i(t)[\omega_j^\mu+s_j(t)\xi_j^\mu]}\Big\rangle_{\mathbf{z}} \\ &= \int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\;\mathrm{e}^{\mathrm{i}\sum_{t\mu}[\hat{w}_t^\mu w_t^\mu+\hat{x}_t^\mu x_t^\mu+w_t^\mu(\sqrt{2}\Omega_\mu+x_t^\mu)]} \\ &\quad\times\int\mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}}\;p_0(\mathbf{q}(0))\;\langle\mathrm{e}^{\frac{-\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{\mu\mathrm{i}}\xi_i^\mu\sum_t[\hat{w}_t^\mu\hat{q}_i(t)+\hat{x}_t^\mu s_i(t)]+\mathrm{i}\sum_{it}\psi_i(t)s_i(t)}\rangle_{\mathbf{z}} \\ &\quad\times\mathrm{e}^{-\mathrm{i}\sum_{ti}\varepsilon_i\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)]}, \end{aligned} \tag{9.13}$$

with $\Omega_\mu = N^{-1/2}\sum_i \omega_i^\mu$, $\omega_i^\mu = (1/2)(R_\mu^{i1} + R_\mu^{i2})$, and $\xi_i^\mu = (1/2)(R_\mu^{i1} - R_\mu^{i2})$, as always. The above expression differs from that of the ordinary batch MG only in the occurrence of the labels ε_i in the exponential of the last line, and in the meaning of the shorthands $\{s_i(t)\}$. This implies that the disorder averages in the present model are fully identical to those of the ordinary batch MG, and we may simply copy expression (4.18), with the standard definitions for the various kernels: $C_{tt'} = N^{-1}\sum_i s_i(t)s_i(t')$, $K_{tt'} = N^{-1}\sum_i s_i(t)\hat{q}_i(t')$, and $L_{tt'} = N^{-1}\sum_i \hat{q}_i(t)\hat{q}_i(t')$. As always, the latter kernels are subsequently isolated via δ-functions written in integral representation, which then generates conjugate kernels. We put $\mathcal{D}C = \prod_{tt'}[dC_{tt'}/\sqrt{2\pi}]$, $\mathcal{D}K = \prod_{tt'}[dK_{tt'}/\sqrt{2\pi}]$ and $\mathcal{D}L = \prod_{tt'}[dL_{tt'}/\sqrt{2\pi}]$ (with similar definitions for $\mathcal{D}\hat{C}$, $\mathcal{D}\hat{K}$ and $\mathcal{D}\hat{L}$), and find (4.22) being generalized to

$$\begin{aligned} \overline{Z[\boldsymbol{\psi}]} &= \int[\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\;\mathrm{e}^{\mathrm{i}N\sum_{tt'}[\hat{C}_{tt'}C_{tt'}+\hat{K}_{tt'}K_{tt'}+\hat{L}_{tt'}L_{tt'}]+\mathcal{O}(\log(N))} \\ &\quad\times\int\mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\;\mathrm{e}^{\mathrm{i}\sum_{t\mu}[\hat{w}_t^\mu w_t^\mu+\hat{x}_t^\mu x_t^\mu+w_t^\mu x_t^\mu]} \\ &\quad\times\mathrm{e}^{-(1/2)\sum_{\mu tt'}\left[w_t^\mu w_{t'}^\mu+\hat{w}_t^\mu L_{tt'}\hat{w}_{t'}^\mu+2\hat{x}_t^\mu K_{tt'}\hat{w}_{t'}^\mu+\hat{x}_t^\mu C_{tt'}\hat{x}_{t'}^\mu\right]} \end{aligned}$$

$$\times \int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}}\, p_0(\mathbf{q}(0))\, \langle e^{-i\sum_{ti}\varepsilon_i\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)]+i\sum_{it}\psi_i(t)s_i(t)}$$
$$\times\, e^{-i\sum_i\sum_{tt'}[\hat{C}_{tt'}s_i(t)s_i(t')+\hat{K}_{tt'}s_i(t)\hat{q}_i(t')+\hat{L}_{tt'}\hat{q}_i(t)\hat{q}_i(t')]}\rangle_{\mathbf{z}}. \tag{9.14}$$

For initial conditions of the form $p_0(\mathbf{q}) = \prod_i p_0(q_i)$, the i-dependent terms in the disorder-averaged generating functional (9.14) again factorize fully over the agents i. If we also carry out the final transformation $\hat{q}_i(t) \to -\varepsilon_i\hat{q}_i(t)$ in (9.14), we arrive at

$$\overline{Z[\boldsymbol{\psi}]} = \int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\, e^{N[\Psi+\Phi+\Omega]+\mathcal{O}(\log(N))} \tag{9.15}$$

with

$$\Psi = i\sum_{tt'}[\hat{C}_{tt'}C_{tt'} + \hat{K}_{tt'}K_{tt'} + \hat{L}_{tt'}L_{tt'}] \tag{9.16}$$

$$\Phi = \alpha\log\Bigg[\int \mathcal{D}w\mathcal{D}\hat{w}\mathcal{D}x\mathcal{D}\hat{x}\, e^{i\sum_t[\hat{w}_t w_t+\hat{x}_t x_t+w_t x_t]}$$
$$\times\, e^{-(1/2)\sum_{tt'}[w_t w_{t'}+\hat{w}_t L_{tt'}\hat{w}_{t'}+2\hat{x}_t K_{tt'}\hat{w}_{t'}+\hat{x}_t C_{tt'}\hat{x}_{t'}]}\Bigg], \tag{9.17}$$

$$\Omega = \frac{1}{N}\sum_i \log\Big\langle \int \mathcal{D}q\mathcal{D}\hat{q}\, p_0(q(0))$$
$$\times\, e^{i\sum_t[\hat{q}(t)[q(t+1)-q(t)-\theta_i(t)]+\psi_i(t)\sigma[q(t),z(t)]]-i\sum_{tt'}\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')}$$
$$\times\, e^{-i\sum_{tt'}[\hat{C}_{tt'}\sigma[q(t),z(t)|T_i]\sigma[q(t'),z(t')|T_i]-\varepsilon_i\hat{K}_{tt'}\sigma[q(t),z(t)|T_i]\hat{q}(t')]}\Big\rangle_{\mathbf{z}}. \tag{9.18}$$

The sub-dominant $\mathcal{O}(\log(N))$ term in the exponent of (9.15) is independent of the generating fields $\{\psi_i(t), \theta_i(t)\}$, and the decision noise averages have as always been reduced to single-site (but multiple-time) ones: $\langle g[z(1), z(2), \ldots]\rangle_{\mathbf{z}} = \int\prod_t [\mathrm{d}z(t)P(z(t))]\; g[z(1), z(2), \ldots]$. Thus, compared with the standard batch MG of Chapter 4, the generalization of our model to include diversity and 'trend followers' has affected only the function Ω.

The integral (9.15) is evaluated by steepest descent, leading to saddle-point equations that are nearly identical to those of the ordinary batch MG. Extremization of the exponent $N[\Psi + \Phi + \Omega]$ of with respect to $\{C, \hat{C}, K, \hat{K}, L, \hat{L}\}$ gives the saddle-point equations

$$C_{tt'} = \langle\, \sigma[q(t), z(t)|T]\; \sigma[q(t'), z(t')|T]\,\rangle_\star, \tag{9.19}$$

$$K_{tt'} = -\langle\, \varepsilon\sigma[q(t), z(t)|T]\; \hat{q}(t')\,\rangle_\star, \tag{9.20}$$

$$L_{tt'} = \langle\, \hat{q}(t)\hat{q}(t')\,\rangle_\star, \tag{9.21}$$

$$\hat{C}_{tt'} = \frac{i\partial\Phi}{\partial C_{tt'}}, \quad \hat{K}_{tt'} = \frac{i\partial\Phi}{\partial K_{tt'}}, \quad \hat{L}_{tt'} = \frac{i\partial\Phi}{\partial L_{tt'}}, \tag{9.22}$$

where

$$\langle g[\{q,\hat{q},z\}|\varepsilon,T]\rangle_\star = \lim_{N\to\infty}\frac{1}{N}\sum_i\left\{\frac{\int\mathcal{D}q\mathcal{D}\hat{q}\;\langle M_i[\{q,\hat{q},z\}|\varepsilon,T]g[\{q,\hat{q},z\}|\varepsilon,T]\rangle_{\mathbf{z}}}{\int\mathcal{D}q\mathcal{D}\hat{q}\;\langle M_i[\{q,\hat{q},z\}|\varepsilon,T]\rangle_{\mathbf{z}}}\right\} \tag{9.23}$$

$$\begin{aligned} M_i[\{q,\hat{q},z\}|\varepsilon,T] = p_0(q(0))\; &\mathrm{e}^{\mathrm{i}\sum_t\hat{q}(t)[q(t+1)-q(t)-\theta_i(t)]+i\sum_t\psi_i(t)\sigma[q(t),z(t)|T_i]} \\ &\times \mathrm{e}^{-\mathrm{i}\sum_{tt'}[\hat{C}_{tt'}\sigma[q(t),z(t)|T_i]\sigma[q(t'),z(t')|T_i]-\varepsilon\hat{K}_{tt'}\sigma[q(t),z(t)|T_i]\hat{q}(t')]} \\ &\times \mathrm{e}^{-\mathrm{i}\sum_{tt'}\hat{L}_{tt'}\hat{q}(t)\hat{q}(t')}. \end{aligned} \tag{9.24}$$

9.2.3 The saddle-point equations

One now proceeds to simplify the saddle-point equations, following the established route. However, we will start to encounter subtle differences with the simple batch MG, due to the presence of the trend followers. Upon taking derivatives of the generating functional (9.12) with respect to the fields $\{\psi_i(t),\theta_i(t)\}$ (after which we may put[61] $\psi_i(t)\to 0$ and $\theta_i(t)\to\theta_a(t)-\epsilon_i\theta_b(t)$ for all i and t) one finds that the physical meaning of the order parameters $\{C,G\}$ is indeed that given by equations (9.8) and (9.9), and that

$$L_{tt'} = 0, \quad K_{tt'} = i\hat{G}_{tt'}. \tag{9.25}$$

These relations, together with the causality of G and $\hat{G}$, imply that the evaluation of Φ carried out in Chapter 4 continues to hold in the present model, so equation (4.44) is again valid provided we replace G by $\hat{G}$, which gives

$$\Phi = -\alpha\mathrm{Tr}\log(\mathbf{1}+\hat{G}) - \frac{1}{2}\alpha\sum_{tt'}L_{tt'}[(\mathbf{1}+\hat{G})^\dagger D^{-1}(\mathbf{1}+\hat{G})]^{-1}_{tt'} + \mathcal{O}(L^2). \tag{9.26}$$

Expression (9.26) allows us to calculate the remaining saddle-point equations

$$\hat{C}_{tt'} = 0, \tag{9.27}$$

$$\hat{K}_{tt'} = -\alpha(\mathbf{1}+\hat{G})^{-1}_{t't}, \tag{9.28}$$

$$\hat{L}_{tt'} = -\frac{1}{2}i\alpha[(\mathbf{1}+\hat{G})^{-1}D(\mathbf{1}+\hat{G}^\dagger)^{-1}]_{tt'}, \tag{9.29}$$

where $D_{tt'} = 1 + C_{tt'}$. These results simplify our saddle-point equations to a set of closed equations for C and $\hat{G}$, the solution of which is used to calculate G:

$$C_{tt'} = \langle\;\sigma[q(t),z(t)|T]\;\sigma[q(t'),z(t')|T]\;\rangle_\star, \tag{9.30}$$

[61] The reason for choosing this specific type of external field $\theta_i(t)$ is that the kernel K will in the present model no longer be proportional to the response function G. Whereas G can ultimately be written in the usual manner as a derivative with respect to the field $\theta_a(t)$, we need the new field $\theta_b(t)$ to do the same for the kernel K.

$$G_{tt'} = -i\langle\, \sigma[q(t), z(t)|T]\; \hat{q}(t')\,\rangle_\star, \tag{9.31}$$

$$\hat{G}_{tt'} = i\langle\, \varepsilon\, \sigma[q(t), z(t)|T]\; \hat{q}(t')\,\rangle_\star, \tag{9.32}$$

with the measure

$$\langle g[\ldots]\rangle_\star = \sum_{\varepsilon=\pm 1} \int \mathrm{d}T\; W(\varepsilon, T) \left\{ \frac{\int \mathcal{D}q\mathcal{D}\hat{q}\; \langle M[\{q,\hat{q},z\}|\varepsilon, T] g[\ldots]\rangle_{\mathbf{z}}}{\int \mathcal{D}q\mathcal{D}\hat{q}\; \langle M[\{q,\hat{q},z\}|\varepsilon, T]\rangle_{\mathbf{z}}} \right\}, \tag{9.33}$$

$$\begin{aligned} M[\{q,\hat{q},z\}|\varepsilon, T] &= p_0(q(0)) \\ &\quad \times \mathrm{e}^{\mathrm{i}\sum_t \hat{q}(t)[q(t+1)-q(t)-\theta_a(t)+\varepsilon\theta_b(t)-\alpha\varepsilon\sum_{t'}(\mathbb{1}+\hat{G})^{-1}_{tt'}\sigma[q(t'),z(t')|T]]} \\ &\quad \times \mathrm{e}^{-(1/2)\alpha\sum_{tt'}\hat{q}(t)[(\mathbb{1}+\hat{G})^{-1}D(\mathbb{1}+\hat{G}^\dagger)^{-1}]_{tt'}\hat{q}(t')}. \end{aligned} \tag{9.34}$$

Thus, in the present model the traditional role of G as an order parameter kernel has been taken over by $\hat{G}$, unless we choose $W(\varepsilon, T) = \delta_{\varepsilon,-1}W(T)$, in which case the definitions of G and $\hat{G}$ simply become identical.

9.2.4 The effective single-agent equation

Our final simplification consists of the usual elimination of the integration variables $\{\hat{q}\}$, using causality. The argument is identical to that in Chapter 4, with only the kernel G replaced by $\hat{G}$ (note that in Chapter 4 we did not require the values of G, only the fact that $G_{tt'} = 0$ for $t \leq t'$, which is similarly true for $\hat{G}$). Thus we find once more that our measure $M[\ldots]$ is normalized,

$$\int \mathcal{D}q\mathcal{D}\hat{q}\; M[\{q,\hat{q},z\}|\varepsilon, T] = \int \mathrm{d}q(0)\; p_0(q(0)) = 1 \tag{9.35}$$

as well as the following two relations

$$\langle \sigma[q(t), z(t)|T]\hat{q}(t')\rangle_\star = i\frac{\partial}{\partial\theta_a(t')}\langle\sigma[q(t), z(t)|T]\rangle_\star, \tag{9.36}$$

$$\langle \varepsilon\sigma[q(t), z(t)|T]\hat{q}(t')\rangle_\star = -i\frac{\partial}{\partial\theta_b(t')}\langle\sigma[q(t), z(t)|T]\rangle_\star. \tag{9.37}$$

We may now integrate out the variables $\{\hat{q}\}$, and write our saddle-point equations in the form

$$C_{tt'} = \langle \sigma[q(t), z(t)|T]\sigma[q(t'), z(t')|T]\rangle_\star, \tag{9.38}$$

$$\hat{G}_{tt'} = \frac{\partial}{\partial \theta_b(t')} \langle \sigma[q(t), z(t)|T] \rangle_\star, \tag{9.39}$$

with

$$\langle g[\{q,z\}|T] \rangle_\star = \sum_{\varepsilon=\pm 1} \int_0^\infty \mathrm{d}T \, W(\varepsilon, T) \int \Big[\prod_t \mathrm{d}q(t) \Big] \langle M[\{q,z\}|\varepsilon, T] g[\{q,z\}|T] \rangle_{\mathbf{z}}, \tag{9.40}$$

$$\begin{aligned} M[\{q,z\}|\varepsilon, T] &= p_0(q(0)) \int \prod_t \Big[\frac{\mathrm{d}\hat{q}(t)}{2\pi} \Big] \mathrm{e}^{-(1/2)\alpha \sum_{tt'} \hat{q}(t)[(\mathbf{1}+\hat{G})^{-1} D (\mathbf{1}+\hat{G}^\dagger)^{-1}]_{tt'} \hat{q}(t')} \\ &\quad \times \mathrm{e}^{\mathrm{i} \sum_t \hat{q}(t) \left[q(t+1) - q(t) - \theta_a(t) + \varepsilon \theta_b(t) - \alpha\varepsilon \sum_{t'} (\mathbf{1}+\hat{G})^{-1}_{tt'} \sigma[q(t'), z(t')|T] \right]} \\ &= p_0(q(0)) \int \prod_t \Big[\frac{\mathrm{d}\eta(t)}{\sqrt{2\pi}} \Big] \frac{\mathrm{e}^{-(1/2) \sum_{tt'} \eta(t)[(\mathbf{1}+\hat{G})^{-1} D (\mathbf{1}+\hat{G}^\dagger)^{-1}]_{tt'} \eta(t')}}{\sqrt{\det[(\mathbf{1}+\hat{G})^{-1} D (\mathbf{1}+\hat{G}^\dagger)^{-1}]}} \\ &\quad \times \prod_{t \geq 0} \delta \Big[q(t+1) - q(t) - \theta_a(t) + \varepsilon \theta_b(t) \\ &\quad - \alpha\varepsilon \sum_{t'} (\mathbf{1}+\hat{G})^{-1}_{tt'} \sigma[q(t'), z(t')|T] - \sqrt{\alpha}\eta(t) \Big]. \end{aligned} \tag{9.41}$$

If desired, the response function G follows from the following identity

$$G_{tt'} = \frac{\partial}{\partial \theta_a(t')} \langle \sigma[q(t), z(t)|T] \rangle_\star. \tag{9.42}$$

In fact, we no longer need G, so for the purpose of finding the simplest exact theory we might from now on as well put $\theta_a(t) \to 0$ and $\theta_b(t) \to \tilde{\theta}(t)$, and concentrate solely on the fundamental pair $\{C, \hat{G}\}$.

We then recognize that the measure (9.41), which is here parametrized by the values of (ε, T), describes a single-agent process of the form

$$q(t+1) = q(t) - \varepsilon\tilde{\theta}(t) + \alpha\varepsilon \sum_{t' \leq t} (\mathbf{1}+\hat{G})^{-1}_{tt'} \sigma[q(t'), z(t')|T] + \sqrt{\alpha}\, \eta(t) \tag{9.43}$$

in which $\eta(t)$ is a Gaussian noise, with zero mean and with temporal correlations given by $\langle \eta(t)\eta(t') \rangle = \Sigma_{tt'}$:

$$\Sigma = (\mathbf{1}+\hat{G})^{-1} D (\mathbf{1}+\hat{G}^\dagger)^{-1} \tag{9.44}$$

Causality ensures that $\hat{G}_{tt'} = 0$ for all $t' \geq t$, so $(\mathbf{1}+\hat{G})^{-1}_{tt'} = \sum_{n \geq 0} (-1)^n [\hat{G}^n]_{tt'} = 0$ for $t' > t$. The correlation and response functions $\{C, \hat{G}\}$ are the dynamic order parameters of the problem, to be solved self-consistently from the closed equations

(9.38) and (9.39), in which $\langle \ldots \rangle_\star$ now denotes averaging over the effective single-agent process (9.43) *for a given choice of* (ε, T) and over the zero-average Gaussian noise variables $\{z(t)\}$ with $\langle z(t)z(t')\rangle = \delta_{tt'}$, followed by further averaging over the distribution $W(\varepsilon, T)$.

9.2.5 Overall bid statistics and volatility

Also the calculation of the volatility matrix is found to require only minor modifications to the corresponding calculation in Chapter 4. We define the generating functional for overall bid statistics

$$Z[\boldsymbol{\phi}] = \langle \mathrm{e}^{\mathrm{i}\sqrt{2}\sum_{t\geq 0}\sum_\mu \phi_\mu(t) A^\mu[\mathbf{q}(t),\mathbf{z}(t)]}\rangle. \tag{9.45}$$

This generator is obtained from the previous generating functional (9.12) by the substitution

$$\mathrm{e}^{\mathrm{i}\sum_{it}\psi_i(t)s_i(t)} \;\rightarrow\; \mathrm{e}^{\mathrm{i}\sum_{t\mu}\phi_\mu(t)[\sqrt{2}\Omega_\mu + x_t^\mu]}.$$

We now find the previous expressions (4.60) and (4.61) for the simple batch model being generalized to

$$\begin{aligned}
\overline{Z[\boldsymbol{\phi}]} &= \int \mathcal{D}\mathbf{w}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{x}}\; \mathrm{e}^{\mathrm{i}\sum_{t\mu}[\hat{w}_t^\mu[w_t^\mu - \phi_\mu(t)] + \hat{x}_t^\mu x_t^\mu + w_t^\mu(\sqrt{2}\Omega_\mu + x_t^\mu)]} \\
&\quad \times \int \mathcal{D}\mathbf{q}\mathcal{D}\hat{\mathbf{q}}\; p_0(\mathbf{q}(0))\; \langle \overline{\mathrm{e}^{(-\mathrm{i}\sqrt{2}/\sqrt{N})\sum_{\mu i}\xi_i^\mu \sum_t [\hat{w}_t^\mu \hat{q}_i(t) + \hat{x}_t^\mu s_i(t)]}}\rangle_{\mathbf{z}} \\
&\quad \times \mathrm{e}^{-\mathrm{i}\sum_{ti}\varepsilon_i \hat{q}_i(t)[q_i(t+1) - q_i(t) - \theta_i(t)]} \\
&= \int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\; \mathrm{e}^{N[\Psi+\Phi+\Omega] + \mathcal{O}(N^0)}
\end{aligned} \tag{9.46}$$

with, after the further transformation $\hat{q}_i(t) \rightarrow -\varepsilon_i \hat{q}_i(t)$ and after restoration of the meaning of the variables $s_i(t) = \sigma[q_i(t), z_i(t)|T_i]$:

$$\Psi = i\sum_{tt'}[\hat{C}_{tt'}C_{tt'} + \hat{K}_{tt'}K_{tt'} + \hat{L}_{tt'}L_{tt'}], \tag{9.47}$$

$$\begin{aligned}
\Phi &= \frac{1}{N}\sum_\mu \log\Bigg[\int \mathcal{D}w\mathcal{D}\hat{w}\mathcal{D}x\mathcal{D}\hat{x}\; \mathrm{e}^{i\sum_t [\hat{w}_t[w_t - \phi_\mu(t)] + \hat{x}_t x_t + w_t x_t]} \\
&\quad \times \mathrm{e}^{-(1/2)\sum_{tt'}[w_t w_{t'} + \hat{w}_t L_{tt'}\hat{w}_{t'} + 2\hat{x}_t K_{tt'}\hat{w}_{t'} + \hat{x}_t C_{tt'}\hat{x}_{t'}]}\Bigg],
\end{aligned} \tag{9.48}$$

$$\begin{aligned}
\Omega &= \frac{1}{N}\sum_i \log\Big\langle \int \mathcal{D}q\mathcal{D}\hat{q}\; p_0(q(0)) \\
&\quad \times \mathrm{e}^{\mathrm{i}\sum_t \hat{q}(t)[q(t+1) - q(t) - \theta_i(t)] - i\sum_{tt'}\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')}
\end{aligned}$$

$$\times\ \left\langle e^{-\mathrm{i}\sum_{tt'}[\hat{C}_{tt'}\sigma[q(t),z(t)|T_i]\sigma[q(t'),z(t')|T_i]-\varepsilon_i\hat{K}_{tt'}\sigma[q(t),z(t)|T_i]\hat{q}(t')]}\right\rangle_{\mathbf{z}}. \quad (9.49)$$

All the remaining manipulations in the derivation of the bid statistics involve the function Φ only, whereas the differences between the simple batch MG and the present generalized model are concentrated in the function Ω. Thus, we are allowed to copy the final result of the calculation in Chapter 4, and replace in the latter simply $G \to \hat{G}$ (reflecting the different values taken by the entries of the kernel K in the present model). We then find that the disorder-averaged overall bid is zero on average at any time, and that the entries of the volatility matrix $\overline{\Xi}$ are given by

$$\begin{aligned}\overline{\Xi}_{tt'} &= \lim_{N\to\infty}\overline{\langle A^{\mu}[\mathbf{q}(t),\mathbf{z}(t)]A^{\nu}[\mathbf{q}(t'),\mathbf{z}(t')]\rangle}\\ &= \frac{1}{2}\delta_{\mu\nu}[(\mathbb{1}+\hat{G})^{-1}D(\mathbb{1}+\hat{G}^{\dagger})^{-1}]_{tt'} = \frac{1}{2}\Sigma_{tt'}.\end{aligned} \quad (9.50)$$

Thus the noise term $\eta(t)$ in the effective single agent process (9.43) here continues to represent the overall market fluctuations.

9.2.6 TTI stationary states

Having derived the effective single-agent description for the present process, which defines implicit closed dynamic equations for the order parameter kernels $\{C, \hat{G}\}$, we proceed to the calculation of those stationary solutions of this process which are fully TTI and do not exhibit anomalous response. Thus we make the standard ansatz $C_{tt'} = C(t-t')$ and $\hat{G}_{tt'} = \hat{G}(t-t')$, with $\hat{\chi} = \sum_{t\geq 0}\hat{G}(t)$ finite. Here we have to be somewhat more careful, in that the effective single-agent equation (9.43) is to be solved at first for arbitrary choices of the parameters (ε, T), after which the persistent order parameters are to be calculated as averages over all such choices, weighted by the distribution $W(\varepsilon, T)$.

Upon writing (9.43) in integrated form, and after transformation of the variable $q(t)$ according to $\tilde{q}(t) = q(t)/t$ we obtain for a given choice of (ε, T):

$$\tilde{q}(t) = \frac{q(0)}{t} + \frac{\sqrt{\alpha}}{t}\sum_{t'=0}^{t-1}\left[\eta(t') - \frac{\varepsilon}{\sqrt{\alpha}}\tilde{\theta}(t')\right] + \frac{\alpha\varepsilon}{t}\sum_{t'=0}^{t-1}\sum_{s\geq 0}(\mathbb{1}+\hat{G})^{-1}_{t's}\sigma[s\tilde{q}(s), z(s)|T]. \quad (9.51)$$

Upon taking the limit $t\to\infty$, assuming $\tilde{q} = \lim_{t\to\infty}\tilde{q}(t)$ to exist and $\hat{\chi}$ being finite, we find in the usual manner that

$$\tilde{q} = \sqrt{\alpha}\overline{\eta} - \varepsilon\overline{\theta} + \frac{\alpha\varepsilon\overline{\sigma}}{1+\hat{\chi}} \quad (9.52)$$

with $\overline{\eta} = \lim_{\tau\to\infty} \tau^{-1}\sum_{t\leq\tau}\eta(t)$, with $\overline{\theta} = \lim_{\tau\to\infty} \tau^{-1}\sum_{t\leq\tau}\tilde{\theta}(t)$, with $\overline{\sigma} = \lim_{\tau\to\infty} \tau^{-1}\sum_{t\leq\tau}\sigma[\tilde{q}t|T]$, and with $\sigma[q|T] = \int dz\ P(z)\sigma[q,z|T]$. Given the anti-symmetry of the function $\sigma[q|T]$, we must demand that $\text{sgn}[\tilde{q}] = \text{sgn}[\overline{\sigma}]$ for any (ε, T). Furthermore, upon using the simple relation

$$\begin{aligned}\hat{\chi} &= \sum_{t\geq 0}\frac{\partial}{\partial\tilde{\theta}(t)}\sum_{\varepsilon=\pm1}\int_0^\infty dT\ W(\varepsilon,T)\int d\overline{\eta}\ P(\overline{\eta})\overline{\sigma}\Big|_{\varepsilon,T} \\ &= \frac{\partial}{\partial\overline{\theta}}\sum_{\varepsilon=\pm1}\int_0^\infty dT\ W(\varepsilon,T)\int d\overline{\eta}\ P(\overline{\eta})\overline{\sigma}\Big|_{\varepsilon,T} \\ &= -\sum_{\varepsilon=\pm1}\int_0^\infty dT\ W(\varepsilon,T)\frac{\varepsilon}{\sqrt{\alpha}}\int d\overline{\eta}\ P(\overline{\eta})\frac{\partial}{\partial\overline{\eta}}\overline{\sigma}\Big|_{\varepsilon,T}, \end{aligned} \tag{9.53}$$

we no longer need the persistent external field $\overline{\theta}$ for the calculation of the order parameter $\hat{\chi}$. Its removal simplifies relation (9.52) to

$$\tilde{q} = \sqrt{\alpha}\overline{\eta} + \frac{\alpha\varepsilon\overline{\sigma}}{1+\hat{\chi}}. \tag{9.54}$$

In view of our experience with the ordinary batch MG we make the ansatz that $1+\hat{\chi} > 0$, which will be confirmed a posteriori. However, in following our usual strategy, i.e. expressing the solution $\overline{\sigma}$ of equation (9.52) in terms of the time-averaged Gaussian noise $\overline{\eta}$, followed by a self-consistent calculation of the persistent order parameters $\{c, \phi, \hat{\chi}\}$, we now encounter a new problem. Whereas the solution $\overline{\sigma}$ is unique for $\varepsilon = -1$ (i.e. for the minority seeking agents), this is no longer true when $\varepsilon = 1$ (i.e. for the trend followers). Again we distinguish between 'fickle' trajectories, where $\tilde{q} = 0$, and 'frozen' transjectories, where $\tilde{q} \neq 0$ and $\overline{\sigma} = \text{sgn}[\tilde{q}]\sigma[\infty|T]$:

- *Minority seekers*: $\varepsilon = -1$
  Here the situation is exactly as in the ordinary batch MG, with χ being replaced by $\hat{\chi}$. We find two complementary conditions for the existence of the fickle versus the frozen solutions, so $\overline{\sigma}$ is a well-defined function of $\overline{\eta}$:

$$|\overline{\eta}| \leq \frac{\sigma[\infty|T]\sqrt{\alpha}}{1+\hat{\chi}}: \quad \text{'fickle' solution}: \quad \overline{\sigma} = \frac{(1+\hat{\chi})\overline{\eta}}{\sqrt{\alpha}}, \tag{9.55}$$

$$|\overline{\eta}| > \frac{\sigma[\infty|T]\sqrt{\alpha}}{1+\hat{\chi}}: \quad \text{'frozen' solution}: \quad \overline{\sigma} = \sigma[\infty|T]\text{sgn}[\overline{\eta}]. \tag{9.56}$$

- *Majority seekers*: $\varepsilon = 1$
  Here the situation is unfortunately quite different. Only for large values of $|\overline{\eta}|$ will there be a unique (frozen) solution $\overline{\sigma}$. For small values of $|\overline{\eta}|$ we find no

less than three possible solutions, two of which are frozen, with the third being a fickle state

$$|\overline{\eta}| \leq \frac{\sigma[\infty|T]\sqrt{\alpha}}{1+\hat{\chi}}: \quad \text{'fickle' solution}: \quad \overline{\sigma} = -\frac{(1+\hat{\chi})\overline{\eta}}{\sqrt{\alpha}}, \tag{9.57}$$

$$\text{'frozen' solutions}: \quad \overline{\sigma} = \pm\sigma[\infty|T], \tag{9.58}$$

$$|\overline{\eta}| > \frac{\sigma[\infty|T]\sqrt{\alpha}}{1+\hat{\chi}}: \quad \text{'frozen' solution}: \quad \overline{\sigma} = \sigma[\infty|T]\text{sgn}[\overline{\eta}]. \tag{9.59}$$

The above implies that for majority seeking agents we are faced with the problem that the stationary solution will depend on the history of the process, i.e. there will be remanence phenomena. This should not surprise us, in view of the positive feedback in the dynamical equation (9.43) for the effective agent with $\varepsilon = 1$. The situation is illustrated in Fig. 9.2.

The remanence interpretation, however, does tell us how we should regard and deal with the multiple solutions for $\varepsilon = 1$ in the regime $|\overline{\eta}| \leq \sigma[\infty|T]\sqrt{\alpha}/(1+\chi)$. The rational interpretation of these solutions must be that $\overline{\sigma} = \pm\sigma[\infty|T]$ represent coexisting stable states, with the 'fickle' branch (the diagonal line segment in the right-side picture of Fig. 9.2) giving an unstable separatrix. In which of the remaining two stable branches $\overline{\sigma} = \pm\sigma[\infty|T]$ the trend-following agent will ultimately be found, will still depend on the history of the process, but at least we may assume that in the

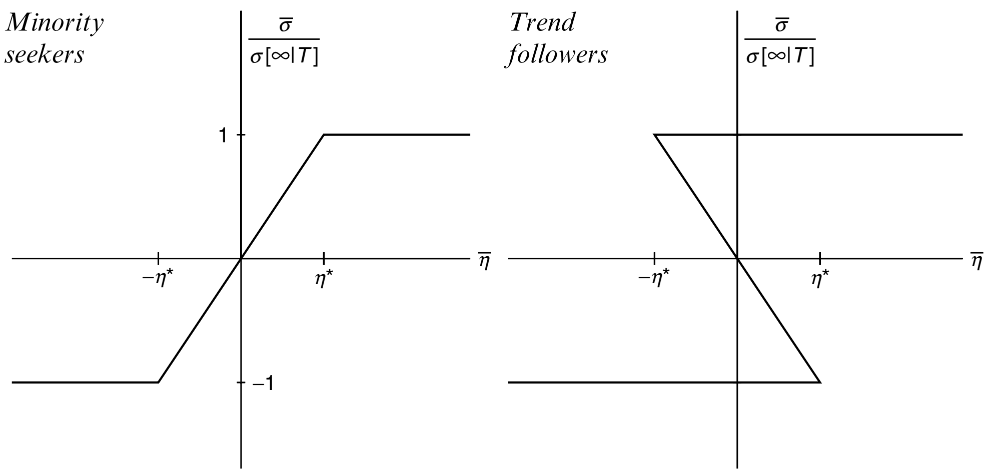


Fig. 9.2 The dependence of the solutions $\overline{\sigma} = \lim_{\tau\to\infty} \tau^{-1}\sum_{t\leq\tau} \sigma[q(t)|T]$ of the stationary state equation (9.54) (thick solid lines) on the long-time average $\overline{\eta} = \lim_{\tau\to\infty} \tau^{-1}\sum_{t\leq\tau} \eta(t)$ of the Gaussian noise in the effective single-agent equation. Left: $\varepsilon = -1$ (minority seeking agents). Right: $\varepsilon = 1$ (majority seeking agents, or 'trend followers'). The cross-over/saturation value $\eta^\star$ is given by $\eta^\star = \sigma[\infty|T]\sqrt{\alpha}/(1+\hat{\chi})$.

regime of coexisting states and normal response the dominant domain of attraction will belong to the solution $\overline{\sigma} = \sigma[\infty|T]\text{sgn}[\overline{\eta}]$. If and when all remanence effects have died down (a situation which, we should be aware, might take a very long time to be achieved), we would then be allowed to put simply[62]

$$\overline{\sigma}(\overline{\eta}) = \sigma[\infty|T]\text{sgn}[\overline{\eta}] \;+\; \delta_{\varepsilon,-1}\,\theta\Big[\frac{\sigma[\infty|T]\sqrt{\alpha}}{1+\hat{\chi}} - |\overline{\eta}|\Big]\Big(\frac{(1+\hat{\chi})\overline{\eta}}{\sqrt{\alpha}} - \sigma[\infty|T]\text{sgn}[\overline{\eta}]\Big). \tag{9.60}$$

Having resolved the issue of expressing $\overline{\sigma}$ in terms of the time-averaged Gaussian noise $\overline{\eta}$ for any choice of (ε, T), we can proceed to the calculation of the persistent order parameters $\{c, \phi, \hat{\chi}\}$, following the by now standard route. Apart from the replacement $\chi \to \hat{\chi}$, the calculation of the variance $\langle \overline{\eta}^2 \rangle$ from the covariance matrix (9.50) is identical to that which led to expression (4.125), so that here we can immediately conclude that

$$\langle \overline{\eta}^2 \rangle = \frac{1+c}{(1+\hat{\chi})^2}. \tag{9.61}$$

Working out the equations for $\{c, \phi, \hat{\chi}\}$, in contrast, will no longer boil down to copying previous results and making simple replacements. Here we will note the impact of having trend followers, for which the relation between $\overline{\sigma}$ and $\overline{\eta}$ differs significantly from that of the minority seekers. As always, our equations will take their simplest form when written in terms of the auxiliary variable

$$v(T) = \frac{\sqrt{\alpha}\,\sigma[\infty|T]}{\sqrt{2(1+c)}}. \tag{9.62}$$

This allows us to put $\overline{\eta} = \langle \overline{\eta}^2 \rangle^{1/2} z$, where z is a zero-average and unit-variance Gaussian variable, and subsequently write equation (9.63) as

$$\overline{\sigma}[z] = \sigma[\infty|T]\left\{\text{sgn}[z] \;+\; \delta_{\varepsilon,-1}\,\theta\Big[v(T)\sqrt{2} - |z|\Big]\Big(\frac{z}{v(T)\sqrt{2}} - \text{sgn}[z]\Big)\right\}. \tag{9.63}$$

We now find

$$c = \sum_{\varepsilon=\pm 1}\int_0^\infty \text{d}T\; W(\varepsilon, T)\; c(\varepsilon, T), \tag{9.64}$$

$$\phi = \sum_{\varepsilon=\pm 1}\int_0^\infty \text{d}T\; W(\varepsilon, T)\; \phi(\varepsilon, T), \tag{9.65}$$

$$\hat{\chi} = \sum_{\varepsilon=\pm 1}\int_0^\infty \text{d}T\; W(\varepsilon, T)\; \hat{\chi}(\varepsilon, T), \tag{9.66}$$

in which the three functions $c(\varepsilon, T)$, $\phi(\varepsilon, T)$, and $\hat{\chi}(\varepsilon, T)$ are to be interpreted as conditional order parameters, describing the macroscopic properties of those agents

[62] This particular resolution of the problem of how to select from the multiple solution options for $\varepsilon = 1$ is equivalent to the so-called Maxwell construction in statistical mechanics.

which are characterized by the individual parameter values (ε, T). These functions are given by the following expressions (where we used (9.53) for the susceptibility)

$$\begin{aligned} c(\varepsilon,T) &= \int Dz\, \overline{\sigma}^2[z] = \delta_{\varepsilon,1}\sigma^2[\infty|T] \\ &\quad +\delta_{\varepsilon,-1}\sigma^2[\infty|T] \int Dz \left\{ \theta\Big[|z| - v(T)\sqrt{2}\Big] + \theta\Big[v(T)\sqrt{2} - |z|\Big] \frac{z^2}{2v^2(T)} \right\} \\ &= \delta_{\varepsilon,1}\sigma^2[\infty|T] \\ &\quad +\delta_{\varepsilon,-1}\sigma^2[\infty|T] \left\{ 1 + \frac{1-2v^2(T)}{2v^2(T)} \mathrm{Erf}[v(T)] - \frac{\mathrm{e}^{-v^2(T)}}{v(T)\sqrt{\pi}} \right\}, \end{aligned} \tag{9.67}$$

$$\begin{aligned} \phi(\varepsilon,T) &= 1 - \delta_{\varepsilon,-1} \int Dz\, \theta\Big[v(T)\sqrt{2} - |z|\Big] \\ &= 1 - \delta_{\varepsilon,-1}\mathrm{Erf}[v(T)], \end{aligned} \tag{9.68}$$

$$\begin{aligned} \hat{\chi}(\varepsilon,T) &= -\frac{\varepsilon}{\sqrt{\alpha}} \int \mathrm{d}\overline{\eta}\; P(\overline{\eta}) \frac{\partial}{\partial \overline{\eta}} \overline{\sigma} = -\frac{\varepsilon(1+\hat{\chi})}{\sqrt{\alpha(1+c)}} \int Dz\, \frac{\mathrm{d}}{\mathrm{d}z}\, \overline{\sigma}[z] \\ &= \delta_{\varepsilon,-1} \frac{(1+\hat{\chi})\sigma[\infty|T]}{v(T)\sqrt{2\alpha(1+c)}} \int Dz\, \theta\Big[v(T)\sqrt{2} - |z|\Big] \\ &\quad -\delta_{\varepsilon,1} \frac{2(1+\hat{\chi})\sigma[\infty|T]}{\sqrt{\alpha(1+c)}} \int Dz\, \delta(z) \\ &= \delta_{\varepsilon,-1} \frac{(1+\hat{\chi})}{\alpha} \Big[1 - \phi(\varepsilon,T)\Big] - \delta_{\varepsilon,1} \frac{\sqrt{2}(1+\hat{\chi})\sigma[\infty|T]}{\sqrt{\alpha\pi(1+c)}}. \end{aligned} \tag{9.69}$$

Insertion of the trio (9.67)–(9.69) into (9.64–9.66) gives us our final closed equations for the persistent order parameters $\{c, \phi, \hat{\chi}\}$. Upon writing

$$W(\varepsilon,T) = p\delta_{\varepsilon,1}W_+(T) + (1-p)\delta_{\varepsilon,-1}W_-(T), \tag{9.70}$$

where $p = \int \mathrm{d}T\, W(1,T)$ and $1-p = \int \mathrm{d}T\, W(1,T)$, these order parameter equations take the following form

$$\begin{aligned} c &= (1-p)\int_0^\infty \mathrm{d}T\, W_-(T)\, \sigma^2[\infty|T] \left\{ 1 + \frac{1-2v^2(T)}{2v^2(T)}\, \mathrm{Erf}[v(T)] - \frac{\mathrm{e}^{-v^2(T)}}{v(T)\sqrt{\pi}} \right\} \\ &\quad + p\int_0^\infty \mathrm{d}T\, W_+(T)\, \sigma^2[\infty|T], \end{aligned} \tag{9.71}$$

$$\phi = 1 - (1-p)\int_0^\infty \mathrm{d}T\, W_-(T)\, \mathrm{Erf}[v(T)], \tag{9.72}$$

$$\hat{\chi} = \frac{1+\hat{\chi}}{\alpha}\left\{(1-p)\int_0^\infty dT\, W_-(T)\text{Erf}[v(T)] - p\int_0^\infty dT\, W_+(T)\frac{2v(T)}{\sqrt{\pi}}\right\}$$
$$= \frac{1+\hat{\chi}}{\alpha}\left\{1-\phi-p\int_0^\infty dT\, W_+(T)\frac{2v(T)}{\sqrt{\pi}}\right\}.$$

so that

$$\hat{\chi} = \frac{1-\phi-p\int_0^\infty dT\, W_+(T)(2v(T)/\sqrt{\pi})}{\alpha-1+\phi+p\int_0^\infty dT\, W_+(T)(2v(T)/\sqrt{\pi})}. \tag{9.73}$$

Together with (9.62), equations (9.71)–(9.73) form an exact and closed set from which we can extract all the relevant information on the statics in the TTI regime. In fact, upon inserting (9.62) into (9.71) one is seen to obtain a closed equation for c only.

We can immediately see that in the case of additive decision noise, where $\sigma[\infty|T] = 1$ for any $T < \infty$, the effects of this noise drop out completely, as was the case in our previous versions of the MG. This is obviously not true for multiplicative noise. For the purposes of solving the statics in the TTI regime and finding the phase diagrams, we may therefore concentrate on multiplicative noise, and regard the models with absent or additive decision noise simply as being described by the solution corresponding to $W_\pm(T) = \delta(T)$.

To determine whether the traditional scenario of having TTI solutions for sufficiently large α has survived our present generalization of the MG, let us first inspect the solution of the above equations for $\alpha \to \infty$. There we find

$$\lim_{\alpha\to\infty} c = p\int_0^\infty dT\, W_+(T)\,\sigma^2[\infty|T], \quad \lim_{\alpha\to\infty}\phi = p, \quad \lim_{\alpha\to\infty}\hat{\chi} = 0. \tag{9.74}$$

We also see that, although $\hat{\chi}$ itself will generally be negative for large α, the quantity $1+\hat{\chi}$ (which we assumed to be positive as our ansatz) is indeed positive in the large α regime

$$1+\hat{\chi} = \frac{\alpha}{\alpha-1+\phi+p\int_0^\infty dT\, W_+(T)(2v(T)/\sqrt{\pi})}. \tag{9.75}$$

The assumption of normal response is seen to be satisfied for $\alpha > \alpha_c$, where α_c is the point where the susceptibility $\hat{\chi}$ diverges, i.e. where

$$\hat{\chi} \uparrow \infty: \;\; 1-\phi = \alpha + \frac{2p}{\sqrt{\pi}}\int_0^\infty dT\, W_+(T)v(T). \tag{9.76}$$

Thus, the traditional scenario has in principle remained intact. The locations of the actual transitions, however, will obviously depend on the various new parameters of the generalized model.

The volatility σ^2 follows from (9.50) and involves as always non-persistent order parameters

$$\sigma^2 = \frac{1}{2}[(\mathbf{1} + \hat{G})^{-1} D (\mathbf{1} + \hat{G}^\dagger)^{-1}](0). \tag{9.77}$$

As in most previous models, it can only be expressed in terms of persistent order parameters if one is willing to pay the price of approximation. The two approximations (4.154) and (4.155) that were developed in Chapter 4 cannot be used in their original forms, since they assumed having minority seeking agents only. However, if one repeats the derivation of (4.154) and (4.155) for the present model, taking care to separate each term into a contribution from minority seekers and one from trend followers, using our population-specific results (9.67)–(9.69), one finds that the various arguments continue to hold and that the only changes required are to make the replacements

$$\chi \to \hat{\chi}, \quad \phi\sigma^2[\infty] \to \sum_{\varepsilon=\pm 1} \int_0^\infty \mathrm{d}T\, W(\varepsilon, T)\phi(\varepsilon, T)\sigma^2[\infty|T]. \tag{9.78}$$

In particular, approximation σ_A^2 remains unchanged. Hence, we obtain

$$\sigma_A^2 = \frac{1+c}{2(1+\hat{\chi})^2} + \frac{1}{2}(1-c), \tag{9.79}$$

$$\sigma_B^2 = \frac{1 + \sum_{\varepsilon=\pm 1} \int \mathrm{d}T\, W(\varepsilon, T)\phi(\varepsilon, T)\sigma^2[\infty|T]}{2(1+\hat{\chi})^2} + \frac{1}{2}\Big[1 - \sum_{\varepsilon=\pm 1} \int \mathrm{d}T\, W(\varepsilon, T)\phi(\varepsilon, T)\sigma^2[\infty|T]\Big]. \tag{9.80}$$

In the non-ergodic regime $\alpha < \alpha_c$ the above theory, based on $\hat{\chi} < \infty$, is obviously no longer valid, and we would in principle have to solve the full dynamical problem (9.38)–(9.43). At most we may hope to find the low volatility solutions in the non-ergodic regime (similar to how this was done for previous MG versions), by demanding $\hat{\chi} = \infty$ and using this relation, via (9.73), to determine the variable v in (9.71) and (9.72).

9.2.7 Phase diagrams and persistent observables

There is an infinite number of possible distributions $W_\pm(T)$ for which we could now produce phase diagrams if we wished. We will here limit ourselves for practical reasons to those of the simple form $W_\pm(T) = \delta[T - T_\pm]$, i.e. those where the decision noise strength depends only on an agent's species (minority seeker or trend follower), and to (Gaussian) multiplicative decision noise of the type (2.13) (where $\sigma[\infty|T] = \lambda(T) = \text{Erf}[1/T\sqrt{2}]$). This simplifies our TTI equations as follows

$$c = (1-p)\lambda^2(T_-)\left\{1 + \frac{1-2v^2}{2v^2}\,\text{Erf}[v] - \frac{\text{e}^{-v^2}}{v\sqrt{\pi}}\right\} + p\,\lambda^2(T_+), \tag{9.81}$$

$$\phi = 1 - (1-p)\text{Erf}[v], \tag{9.82}$$

$$v = \frac{\sqrt{\alpha}\,\lambda(T_-)}{\sqrt{2(1+c)}}. \tag{9.83}$$

The $\hat{\chi} = \infty$ phase transition occurs at the point where

$$\alpha = (1-p)\text{Erf}[v] - p\frac{2v}{\sqrt{\pi}}\frac{\lambda(T_+)}{\lambda(T_-)}. \tag{9.84}$$

We will henceforth simply put $\lambda(T_\pm) = \lambda_\pm$. The transition condition (9.84) can be written in the more convenient form of a transcendental equation for a single variable only. Following in the footsteps of the simple batch MG, we first eliminate c via insertion of (9.83) into (9.81). From the result we eliminate α via (9.84), leaving an equation for $v \geq 0$ only. One then finds that the solution of the following equation defines the critical point v_c, from which α_c is subsequently calculated using (9.84):

$$(1-p)\lambda_-^2\left\{\text{Erf}[v] - 1 + \frac{\text{e}^{-v^2}}{v\sqrt{\pi}}\right\} - \frac{p\lambda_+\lambda_-}{v\sqrt{\pi}} = 1 + p\lambda_+^2. \tag{9.85}$$

One can easily convince oneself that the left-hand side of equation (9.85), when regarded as a function of v, has the following properties

$$(1-p)\lambda_- > p\lambda_+ : \quad \text{LHS}(0) = \infty, \quad \text{LHS}(\infty) = 0$$
$$\frac{\text{d}}{\text{d}v}\text{LHS}(v) < 0 \quad \text{for } v < \sqrt{\log[(1-p)\lambda_-/p\lambda_+]}$$
$$\frac{\text{d}}{\text{d}v}\text{LHS}(v) > 0 \quad \text{for } v > \sqrt{\log[(1-p)\lambda_-/p\lambda_+]}.$$

$$(1-p)\lambda_- < p\lambda_+ : \quad \text{LHS}(0) = -\infty, \quad \text{LHS}(\infty) = 0$$
$$\frac{\text{d}}{\text{d}v}\text{LHS}(v) > 0 \quad \text{for all } v \geq 0.$$

Since the right-hand side of equation (9.85) is non-negative, we conclude that for $(1-p)\lambda_- < p\lambda_+$ (i.e. dominance of the trend followers) there will be no transition;

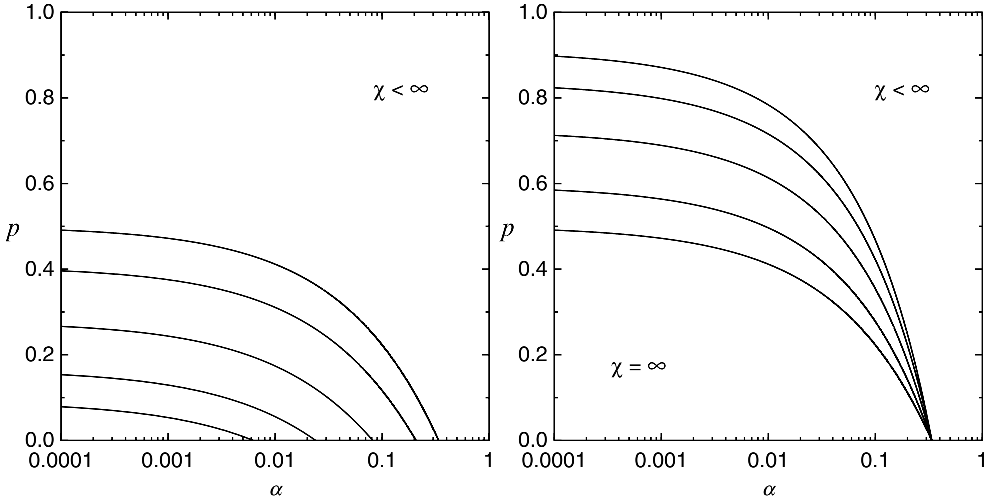


Fig. 9.3 Phase diagram cross-sections of the MG with a fraction p of trend followers, consisting of a high α phase with TTI and $\chi < \infty$ and a low α phase where $\chi = \infty$. Solid curves in left: transition lines for $T_+ = 0$ (deterministic strategy selection by trend followers) and $T_- \in \{0, 1, 2, 4, 8\}$ (multiplicative decision noise for minority seekers), from top to bottom. Solid curves in right: transition lines for $T_- = 0$ (deterministic strategy selection by the minority seekers) and $T_+ \in \{0, 1, 2, 4, 8\}$ (multiplicative decision noise for the trend followers), from bottom to top.

the system will remain in the TTI state for all α. For $(1-p)\lambda_- > p\lambda_+$ (i.e. dominance of the minority seekers) we will have a $\hat{\chi} \to \infty$ transition, but the associated value of α_c is always lower than that found in the absence of trend followers, and obeys $\lim_{(1-p)\lambda_- \downarrow p\lambda_+} \alpha_c = 0$.

Numerical solution of the transcendental equation (9.85) results in phase diagram cross-sections[63] such as those shown in Figs 9.3 and 9.4. We see that the introduction of trend followers in the batch MG, i.e. increasing p, always leads to an enlargement of the ergodic $\chi < \infty$ regime, especially when these trend followers act deterministically. Since α is inversely proportional to the number of agents in the market, this implies that if we increase the fraction of trend followers a *larger* number of agents will be required to eliminate all predictability from the market (since $\overline{\Xi}(\infty) > 0$ in the ergodic regime, and zero in the non-ergodic regime).

The values of the persistent order parameters $\{c, \phi\}$ are readily obtained by numerical solution of equations (9.81)–(9.83). The approximate expressions (9.79) and (9.80) for the volatility here reduce to

[63] Since we still have four control parameters $\{\alpha, p, T_+, T_-\}$ only cross-sections of the control parameter space can be drawn.

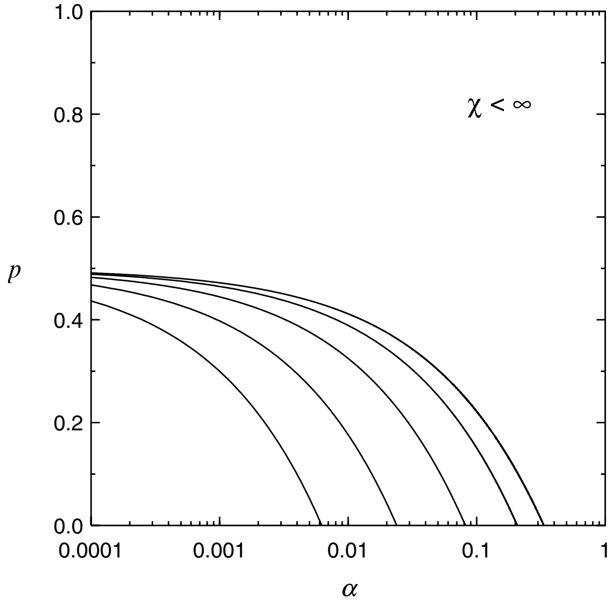


Fig. 9.4 Phase diagrams of cross-sections of the MG with a fraction p of trend followers, consisting of a high α phase with TTI and $\chi < \infty$ and a low α phase where $\chi = \infty$. Solid curves: transition lines for $T_+ = T_- \in \{0, 1, 2, 4, 8\}$ (multiplicative decision noise for all agents), from top to bottom.

$$\sigma_A^2 = \frac{1+c}{2(1+\hat{\chi})^2} + \frac{1}{2}(1-c), \quad \sigma_B^2 = \frac{1+\hat{c}}{2(1+\hat{\chi})^2} + \frac{1}{2}(1-\hat{c}) \tag{9.86}$$

with

$$\hat{c} = p\lambda_+^2 + (1-p)\Big[1 - \mathrm{Erf}(v)\Big]\lambda_-^2. \tag{9.87}$$

Comparison of the resulting theoretical predictions for the present model in TTI stationary states with numerical simulations confirms that there are no further unexpected transitions beyond the one identified already. This can be seen for instance in Figs 9.5 and 9.6 (for the observables $\{c, \phi\}$), and Fig. 9.7 (for the volatility). In the non-ergodic regime $\alpha < \alpha_c(T)$ of the MG the theory has been extended in the usual (ad hoc) manner, by solving our equations on the basis of the premise that $\hat{\chi} = \infty$ remains true all the way down to $\alpha = 0$. The theory is seen to agree perfectly with the experimental data in its regime of validity, in terms of the observables $\{\phi, c\}$, whereas for $\alpha < \alpha_c(T)$ (that is, for those values of p where the non-ergodic phase actually exists) the theory is seen to capture the low volatility solution. The volatility itself, the formula of which we know indeed to be an approximation, shows some deviation between theory and

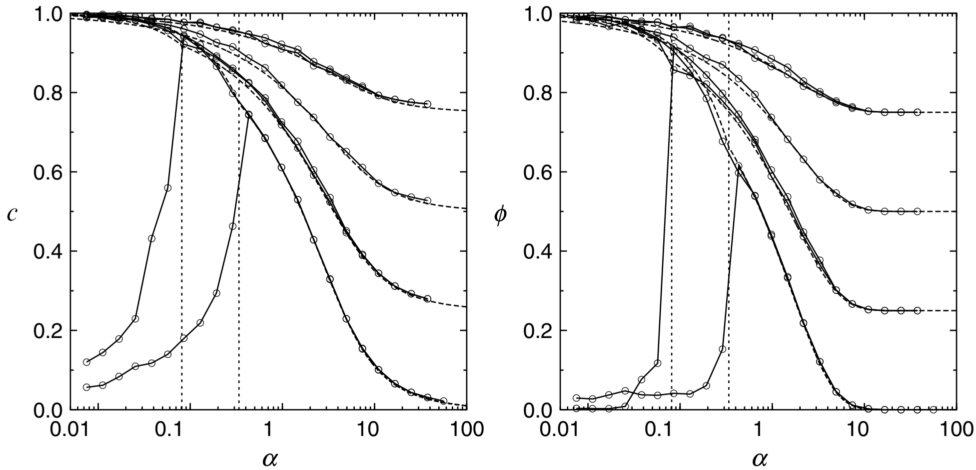


Fig. 9.5 The predicted persistent correlations c (dashed lines, left) and the fraction of frozen agents ϕ (dashed lines, right), together with simulation data (connected markers), for the batch MG with 'fake' histories and trend followers. A fraction p of trend followers play alongside a fraction $1 - p$ of minority seekers, both without decision noise. Results are shown for $p \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, from bottom to top. Initial conditions: $|q_i(0)| = \Delta \in \{0, 10\}$ for all i. The vertical dotted lines mark the $\hat{\chi} = \infty$ transition points for $p \in \{0, \frac{1}{4}\}$ (the theory predicts that there is no such transition for $p \geq \frac{1}{2}$).

numerical data; as for the ordinary batch MG, approximation σ_B^2 is found to be more accurate.

9.2.8 Economic relevance and realism

We know that the batch version of the MG with trend followers will again tend towards Gaussian distributed overall market bids. If the introduction of trend followers is to bring the MG at all closer to the reality of financial time series, at least the on-line version should near its critical line in the (α, p) plane (where one expects instabilities to emerge first) exhibit some of the stylized effects that have so far been absent in previous MG models. However, Fig. 9.8 shows that these stylized effects are still absent, even in small systems (where fluctuations should be most visible). In terms of bid (i.e. price) statistics, there are only quantitative differences between the MG with or without trend followers.

Hence, in economic terms, the main consequences of introducing trend followers into our MG market are therefore the following: they seriously impair the ability of the community of interacting agents to reduce market predictability, and they generally lead to an increase of the volatility. In fact, for the market it would have been much

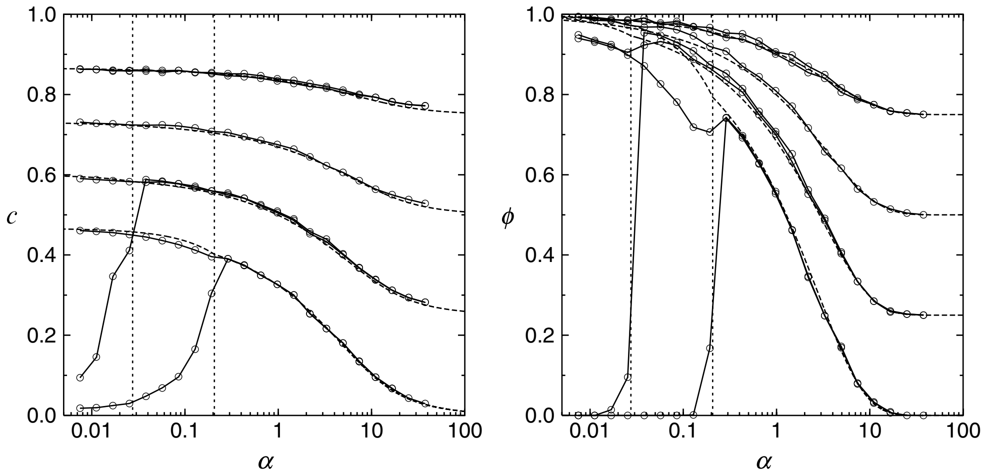


Fig. 9.6 The predicted persistent correlations c (dashed lines, left) and the fraction of frozen agents ϕ (dashed lines, right), together with simulation data (connected markers), for the batch MG with 'fake' histories and trend followers. A fraction p of trend followers without decision noise play alongside a fraction $1-p$ of minority seekers with multiplicative decision noise of strength $T=1$. Results are shown for $p \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, from bottom to top. Initial conditions: $|q_i(0)| = \varDelta \in \{0, 10\}$ for all i. The vertical dotted lines mark the $\hat{\chi} = \infty$ transition points for $p \in \{0, \frac{1}{4}\}$ (the theory predicts that there is no such transition for $p \geq \frac{1}{2}$).

better if the trend followers were to trade purely randomly (i.e. have $T_+ = \infty$) than according to their preferred private strategies. In short: trend followers make our model market definitely less efficient. They can, however, not be accused of causing stylized effects in the MG.

9.3 Producers and speculators

The previous variation on the conventional MG theme was one which did not affect the definition of the (random) strategies. We now turn to a variation that was motivated by the desire to give (some of the) agents the option of not playing at stages of the game where they perceive their chances of winning to be low. Here the strategy definition will change. The idea behind this version of the MG is that it is intuitively clear that in the real world the instability of price time series like those on the stock market is to a large extent related to the self-accelerating effects of speculators pulling out of certain investments, resulting in a price drop that can subsequently cause other speculators to follow suit. In contrast, in the standard MGs everyone has so far been forced to stay and trade in the market, forever. Secondly, giving the agents the option of not trading should generally allow the traded volume to be no longer a fixed number, as it has so far been in the simple versions of the MG.

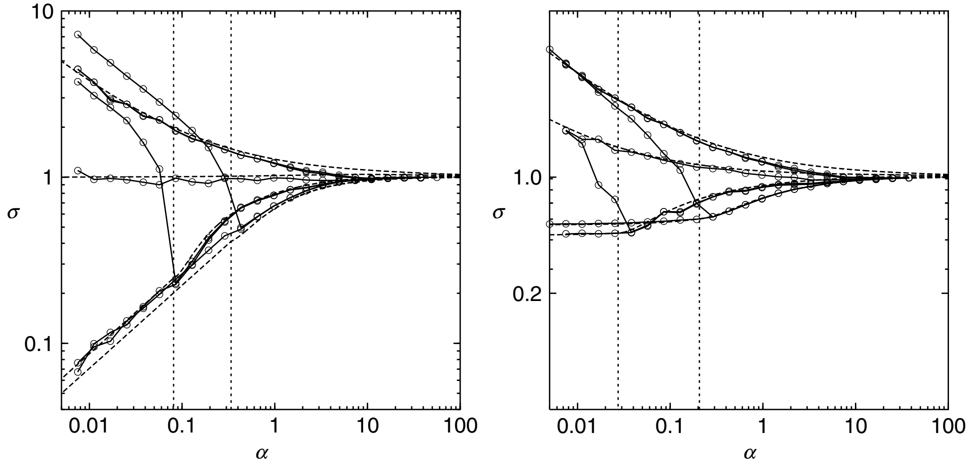


Fig. 9.7 The approximation σ_B (9.86) for the volatility in terms of persistent order parameters only (dotted lines), together with simulation data (connected markers), for the batch MG with fake history and a fraction p of trend followers. Left: no decision noise. Right: multiplicative decision noise of strength $T = 1$ for the minority seekers only. Data are shown for $p \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, from bottom to top in the large α regime. Initial conditions: $|q_i(0)| = \Delta \in \{0, 10\}$ for all i. The vertical dotted lines mark the $\hat{\chi} = \infty$ transition points for $p \in \{0, \frac{1}{4}\}$ (the theory predicts that there is no such transition for $p \geq \frac{1}{2}$).

9.3.1 Definitions

Let us now construct and solve a simple MG model where there are two types of agents: producers and speculators. Producers[64] in the present model are assumed to participate in the market simply to exchange goods. They need the market for a specific vital purpose, which implies that they will therefore not be inclined to withdraw from it, and that they are relatively inflexible in terms of selecting trading strategies. In contrast, speculators will enter and leave the market freely, and only submit a bid if they believe that they will profit from doing so.

The simplest set-up with the above characteristics is one where we have randomly drawn (fake) market histories and where all agents as always have only two strategies, $a \in \{0, 1\}$, but with one of these ($a = 0$) now being the default strategy of not playing, i.e. of submitting the trivial bid $b_i(\ell) = 0$. This leaves for each agent only one non-trivial (randomly drawn) strategy. Thus

$$(\forall \mu \in \{1, \ldots, p\}): \quad R_\mu^{i0} = 0, \quad R_\mu^{i1} \in \{-1, 1\}. \tag{9.88}$$

[64] This particular terminology, which has been adapted from literature, could perhaps be somewhat misleading. Instead of producers it might have been better to speak about producers-consumers, since this particular type of agent will still be allowed to both buy and sell assets. The essence is that they are agents who trade out of necessity. They need the market for exchanging goods, they cannot simply stop trading if they don't like the going prices.

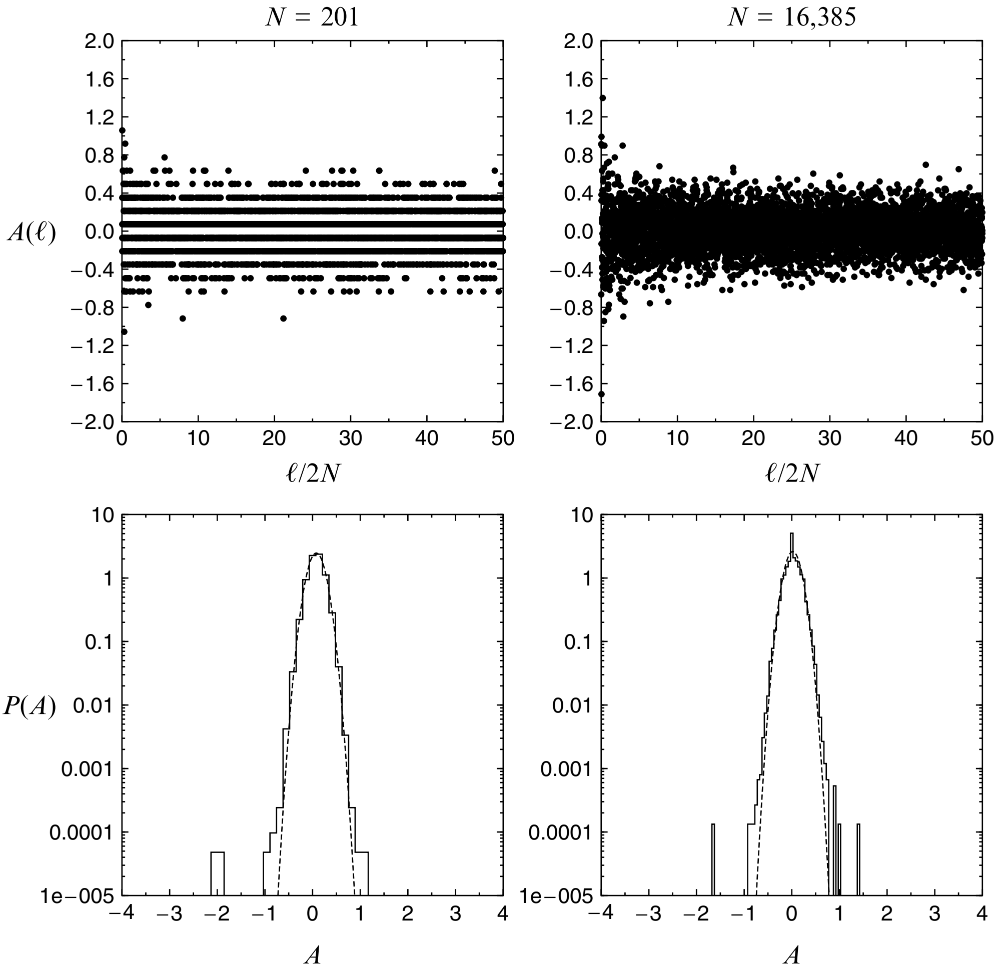


Fig. 9.8 Top: numerical simulation measurements of the re-scaled total bids $A(\ell)$ in the $S = 2$ on-line MG with fake histories and trend followers, operating at criticality: $(\alpha, p) = (0.1, 0.223)$. Bottom: the corresponding distributions $P(A)$ of these observed values (represented as histograms, and plotted logarithmically), together with Gaussian distributions (dashed) of width and average identical to those of the observed $P(A)$. As with the ordinary on-line MG, at the critical point neither small ($N = 201$) nor large systems ($N = 16,385$) exhibit true volatility clustering or fat tails in the bid (i.e. price) distribution.

Each agent i will update the valuation p_{i1} of his non-trivial strategy $a = 1$ in the usual way, i.e. on the basis of how often it would have led to a minority decision. The valuation of the default 'no bid' strategy $a = 0$, however, will be updated according to a constant agent-dependent increment $\varepsilon_i \in \mathbb{R}$ at each update step of the game, which provides us with a mechanism with which to control the appetite or need of individual agents to take part in trading[65]. For on-line versions of this game, we would then end up with the following process

$$p_{i0}(\ell+1) = p_{i0}(\ell) + \varepsilon_i, \tag{9.89}$$

$$p_{i1}(\ell+1) = p_{i1}(\ell) - \frac{\tilde{\eta}}{\sqrt{N}} A(\ell) R^{i1}_{\mu(\ell)}, \tag{9.90}$$

where the pseudo-history integers $\mu(\ell) \in \{1, \ldots, p\}$ are drawn independently at each stage of the game, with uniform probabilities. Apart from the above changes in the strategies and the evolution of their valuations, agents as before seek to be in the minority, and select at each stage of the game the strategy $a_i(\ell) \in \{0, 1\}$ with the highest score, so

$$A(\ell) = \frac{1}{\sqrt{N}} \sum_i b_i(\ell), \quad b_i(\ell) = R^{ia_i(\ell)}_{\mu(\ell)}, \tag{9.91}$$

$$a_i(\ell) = \arg \max_{a \in \{0,1\}} \{p_{ia}(\ell)\}. \tag{9.92}$$

To what extent an agent i operates as a producer or as a speculator will in this model be controlled by the value of ε_i. If $\varepsilon_i > 0$ the agent will only trade if his sole non-trivial trading strategy is sufficiently often profitable (on average); i.e. he would behave like a speculator. If $\varepsilon_i \ll 0$, on the other hand, the agent will nearly always trade and hence behave like a producer.

If we translate the above equations into the usual language of valuation differences, i.e. of $q_i(\ell) = p_{i1}(\ell) - p_{i0}(\ell)$ (here the conventional factor $\frac{1}{2}$ is left out, as it would make equations less tidy), and drop the now obsolete strategy label $a = 1$ by replacing $R^{i1}_\mu \to R^i_\mu$, we find the present equivalent of the conventional 'fake history' on-line MG equations (2.22) and (2.23) to be

$$q_i(\ell+1) = q_i(\ell) - \frac{\tilde{\eta}}{\sqrt{N}} R^{\mu(\ell)}_i A^{\mu(\ell)}[\mathbf{q}(\ell)] - \varepsilon_i, \tag{9.93}$$

$$A^\mu[\mathbf{q}] = \frac{1}{\sqrt{N}} \sum_i \theta[q_i] R^\mu_i, \tag{9.94}$$

(with the step function: $\theta[z > 0] = 1$, $\theta[z < 0] = 0$). In the batch version of this game, updates are made on the basis of the average of the valuation changes over all possible pseudo-histories μ, at discrete time steps $t \in \{0, 1, 2, 3, \ldots\}$. We add the usual external forces $\{\theta_i(t)\}$ to allow for the definition of response functions and adopt

[65] One could also argue that for speculators the parameter ε_i reflects the merit in the alternative option to put their capital in the bank, risk-free, rather than invest it in the market.

the scaling with N of the various terms, which gave the correct scaling in earlier batch models. The simplest equations are obtained upon choosing the learning rate $\tilde{\eta} = 2$ (this only defines the unit of time and is of no further consequence), and we find

$$q_i(t+1) = q_i(t) + \theta_i(t) - \varepsilon_i - \frac{2}{\sqrt{N}} \sum_\mu R_i^\mu A^\mu[\mathbf{q}(t)], \tag{9.95}$$

$$A^\mu[\mathbf{q}] = \frac{1}{\sqrt{N}} \sum_i \theta[q_i] R_i^\mu. \tag{9.96}$$

In the case where $q_i(\ell) = 0$ we assign to $\theta[q_i(\ell)]$ one of the two values $\{0, 1\}$ randomly, with equal probabilities. As always, the observables that we will concentrate on in our analysis, via the generating functional method, are the correlation and response functions, which reduce to

$$C_{tt'} = \lim_{N\to\infty} \frac{1}{N} \sum_i \overline{\langle \text{sgn}[q_i(t)]\text{sgn}[q_i(t')]\rangle}, \tag{9.97}$$

$$G_{tt'} = \lim_{N\to\infty} \frac{1}{N} \sum_i \frac{\partial}{\partial \theta_i(t')} \overline{\langle \text{sgn}[q_i(t)]\rangle}. \tag{9.98}$$

One should be aware that the present definitions imply that random trading no longer corresponds to a volatility value of $\sigma = 1$. This is easily confirmed by explicit calculation of $\langle (A^\mu[\mathbf{q}])^2\rangle$ for randomly and independently drawn $\{q_i\}$ (distributed around zero):

$$\begin{aligned}\sigma^2_{\text{random}} &= \lim_{N\to\infty} \langle (A^\mu[\mathbf{q}])^2\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{ij} R^i_\mu R^j_\mu \langle \theta[q_i]\theta[q_j]\rangle \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_i \langle \theta^2[q_i]\rangle = \frac{1}{2}.\end{aligned} \tag{9.99}$$

9.3.2 The disorder-averaged generating functional

We calculate the generating functional (4.9), which will here no longer involve decision noise, so that the brackets $\langle \ldots \rangle$ only refer to the tie-breaking in cases where $q_i(t) = 0$ (if found necessary)

$$\begin{aligned}\overline{Z[\boldsymbol{\psi}]} &= \langle \mathrm{e}^{\mathrm{i}\sum_{t\geq 0}\sum_i \psi_i(t)\text{sgn}[q_i(t)]}\rangle \\ &= \int p_0(\mathbf{q}(0)) \Big[\prod_{it} \frac{\mathrm{d}q_i(t)\mathrm{d}\hat{q}_i(t)}{2\pi} \mathrm{e}^{i\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)+\varepsilon_i]+i\psi_i(t)\text{sgn}[q_i(t)]}\Big] \\ &\quad \times \overline{\prod_{ij\mu} \mathrm{e}^{\frac{2\mathrm{i}}{N} R_i^\mu \sum_t \hat{q}_i(t)\theta[q_j(t)] R_j^\mu}}.\end{aligned} \tag{9.100}$$

As a direct consequence of having only one non-trivial strategy per agent, the present calculation is in fact much simpler than our previous ones. There is no more any reason to switch to the variables $\{\xi_i^\mu, \omega_i^\mu\}$, and in order to carry out the disorder average we no longer need to introduce both types of auxiliary variables $\{w_t^\mu, x_t^\mu\}$. We now require only $w_t^\mu = N^{-1/2}\sum_i \hat{q}_i(t)R_i^\mu$, to be introduced via δ-functions, with the shorthands $\mathcal{D}\mathbf{w} = \prod_{\mu t}[dw_t^\mu/\sqrt{2\pi}]$, $\mathcal{D}\hat{\mathbf{w}} = \prod_{\mu t}[d\hat{w}_t^\mu/\sqrt{2\pi}]$ (with similar definitions for $\mathcal{D}\mathbf{q}$ and $\mathcal{D}\hat{\mathbf{q}}$):

$$
\begin{aligned}
\overline{Z[\boldsymbol{\psi}]} &= \int \mathcal{D}\mathbf{q}\mathcal{D}\widehat{\mathbf{q}}\, p_0(\mathbf{q}(0))\, \mathrm{e}^{\mathrm{i}\sum_{it}\left\{\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)+\varepsilon_i]+\psi_i(t)\mathrm{sgn}[q_i(t)]\right\}} \\
&\quad\times \int \mathcal{D}\mathbf{w}\mathcal{D}\widehat{\mathbf{w}}\, \mathrm{e}^{\mathrm{i}\sum_{t\mu}\hat{w}_t^\mu w_t^\mu}\, \overline{\prod_{i\mu} \mathrm{e}^{\frac{\mathrm{i}}{\sqrt{N}}R_i^\mu \sum_t \left[2w_t^\mu\theta[q_i(t)]-\hat{w}_t^\mu\hat{q}_i(t)\right]}} \\
&= \int \mathcal{D}\mathbf{q}\mathcal{D}\widehat{\mathbf{q}}\, p_0(\mathbf{q}(0))\, \mathrm{e}^{\mathrm{i}\sum_{it}\left\{\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)+\varepsilon_i]+\psi_i(t)\mathrm{sgn}[q_i(t)]\right\}} \\
&\quad\times \int \mathcal{D}\mathbf{w}\mathcal{D}\widehat{\mathbf{w}}\, \mathrm{e}^{\mathrm{i}\sum_{t\mu}\hat{w}_t^\mu w_t^\mu} \prod_{i\mu} \cos\left[\frac{1}{\sqrt{N}}\sum_t \left[2w_t^\mu\theta[q_i(t)] - \hat{w}_t^\mu\hat{q}_i(t)\right]\right] \\
&= \int \mathcal{D}\mathbf{q}\mathcal{D}\widehat{\mathbf{q}}\, p_0(\mathbf{q}(0))\, \mathrm{e}^{\mathrm{i}\sum_{it}\left\{\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)+\varepsilon_i]+\psi_i(t)\mathrm{sgn}[q_i(t)]\right\}} \\
&\quad\times \int \mathcal{D}\mathbf{w}\mathcal{D}\widehat{\mathbf{w}}\, \mathrm{e}^{\mathrm{i}\sum_{t\mu}\hat{w}_t^\mu w_t^\mu - \frac{1}{2N}\sum_{i\mu}\left(\sum_t\left[2w_t^\mu\theta[q_i(t)]-\hat{w}_t^\mu\hat{q}_i(t)\right]\right)^2 + \mathcal{O}(N^0)}.
\end{aligned}
\tag{9.101}
$$

If we work out the exponent in the last line, we find the conventional two-time observables of the batch MG models, i.e. $C_{tt'} = N^{-1}\sum_i \mathrm{sgn}[q_i(t)]\mathrm{sgn}[q_i(t')]$, $K_{tt'} = N^{-1}\sum_i \mathrm{sgn}[q_i(t)]\hat{q}_i(t')$ and $L_{tt'} = N^{-1}\sum_i \hat{q}_i(t)\hat{q}_i(t')$, but also the two one-time objects $m_t = N^{-1}\sum_i \mathrm{sgn}[q_i(t)]$ and $k_t = N^{-1}\sum_i \hat{q}_i(t)$. All these observable are isolated in the usual manner via appropriate δ-functions. We also use the two relations

$$\frac{1}{N}\sum_i \theta[q_i(t)]\theta[q_i(t')] = \frac{1}{4}(1 + m_t + m_{t'} + C_{tt'}), \tag{9.102}$$

$$\frac{1}{N}\sum_i \theta[q_i(t)]\hat{q}_i(t') = \frac{1}{2}(k_{t'} + K_{tt'}). \tag{9.103}$$

All this results in

$$
\begin{aligned}
\overline{Z[\boldsymbol{\psi}]} &= \int [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}][\mathcal{D}m\mathcal{D}\hat{m}][\mathcal{D}k\mathcal{D}\hat{k}]\, \mathrm{e}^{\mathrm{i}N\sum_t[\hat{m}_t m_t + \hat{k}_t k_t] + \mathcal{O}(\log(N))} \\
&\quad\times \mathrm{e}^{\mathrm{i}N\sum_{tt'}[\hat{C}_{tt'}C_{tt'} + \hat{K}_{tt'}K_{tt'} + \hat{L}_{tt'}L_{tt'}]} \int \mathcal{D}\mathbf{w}\mathcal{D}\widehat{\mathbf{w}}\, \mathrm{e}^{\mathrm{i}\sum_{t\mu}\hat{w}_t^\mu w_t^\mu}
\end{aligned}
$$

$$\times\,\mathrm{e}^{-(1/2)\sum_{\mu tt'}\left[w_t^\mu w_{t'}^\mu(1+m_t+m_{t'}+C_{tt'})+\hat{w}_t^\mu\hat{w}_{t'}^\mu L_{tt'}-2w_t^\mu\hat{w}_{t'}^\mu(k_{t'}+K_{tt'})\right]}$$
$$\times\int\mathcal{D}\mathbf{q}\mathcal{D}\widehat{\mathbf{q}}\,p_0(\mathbf{q}(0))\;\;\mathrm{e}^{\mathrm{i}\sum_{it}\left\{\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)+\varepsilon_i]+\psi_i(t)\mathrm{sgn}[q_i(t)]\right\}}$$
$$\times\,\mathrm{e}^{-\mathrm{i}\sum_i\sum_{tt'}[\hat{C}_{tt'}\mathrm{sgn}[q_i(t)]\mathrm{sgn}[q_i(t')]+\hat{K}_{tt'}\mathrm{sgn}[q_i(t)]\hat{q}_i(t')+\hat{L}_{tt'}\hat{q}_i(t)\hat{q}_i(t')].}$$
$$\times\,\mathrm{e}^{-\mathrm{i}\sum_{it}\left[\hat{m}_t\mathrm{sgn}[q_i(t)]+\hat{k}_t\hat{q}_i(t)\right]}. \tag{9.104}$$

For initial conditions of the form $p_0(\mathbf{q})=\prod_i p_0(q_i)$, we benefit as usual from factorization over the agents i, and obtain the standard form for the generating functional, but now with different definitions for *all three* extensive exponents

$$\overline{Z[\boldsymbol{\psi}]}=\int[\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}][\mathcal{D}m\mathcal{D}\hat{m}][\mathcal{D}k\mathcal{D}\hat{k}]\;\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(\log(N))} \tag{9.105}$$

with

$$\Psi=\mathrm{i}\sum_{tt'}[\hat{C}_{tt'}C_{tt'}+\hat{K}_{tt'}K_{tt'}+\hat{L}_{tt'}L_{tt'}]+\mathrm{i}\sum_t[\hat{m}_t m_t+\hat{k}_t k_t], \tag{9.106}$$

$$\Phi=\alpha\log\Bigg[\int\mathcal{D}w\mathcal{D}\hat{w}\;\mathrm{e}^{-(1/2)\sum_{tt'}[w_t w_{t'}(1+m_t+m_{t'}+C_{tt'})+\hat{w}_t\hat{w}_{t'}L_{tt'}]}$$
$$\times\,\mathrm{e}^{\sum_{tt'}w_t\hat{w}_{t'}(k_{t'}+i\delta_{tt'}+K_{tt'})}\Bigg], \tag{9.107}$$

$$\Omega=\frac{1}{N}\sum_i\log\Bigg[\int\mathcal{D}q\mathcal{D}\hat{q}\;p_0(q(0))$$
$$\times\,\mathrm{e}^{-\mathrm{i}\sum_{tt'}[\hat{C}_{tt'}\mathrm{sgn}[q(t)]\mathrm{sgn}[q(t')]+\hat{K}_{tt'}\mathrm{sgn}[q(t)]\hat{q}(t')]-i\sum_t\left[\hat{m}_t\mathrm{sgn}[q(t)]+\hat{k}_t\hat{q}(t)\right]}$$
$$\times\,\mathrm{e}^{\mathrm{i}\sum_t\left[\hat{q}(t)[q(t+1)-q(t)-\theta_i(t)+\varepsilon_i]+\psi_i(t)\mathrm{sgn}[q(t)]\right]-i\sum_{tt'}\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')}\Bigg]. \tag{9.108}$$

The sub-dominant $\mathcal{O}(\log(N))$ term in the exponent of (9.15) is as always independent of the generating fields $\{\psi_i(t),\theta_i(t)\}$.

The integral (9.105) is evaluated by steepest descent, leading to saddle-point equations from which to solve the various one-time and two-time order parameters[66]. Extremization of the exponent $N[\Psi+\Phi+\Omega]$ of (9.105) with respect to $\{C,\hat{C},K,\hat{K},L,\hat{L},m,\hat{m},k,\hat{k}\}$ gives the saddle-point equations.

$$C_{tt'}=\langle\mathrm{sgn}[q(t)]\mathrm{sgn}[q(t')]\rangle_\star,\quad K_{tt'}=\langle\mathrm{sgn}[q(t)]\hat{q}(t')\rangle_\star, \tag{9.109}$$

[66] In fact we could have defined our observables differently, to be specific: we could have chosen $N^{-1}\sum_i\theta[q_i(t)]\theta[q_i(t')]$ and $N^{-1}\sum_i\theta[q_i(t)]\hat{q}_i(t')$ instead of the present $C_{tt'}$ and $K_{tt'}$, in which case there would not have been a need for $\{m,k\}$. The reasons for the present choice are, firstly, that our theory will now be formulated in terms of exactly the order parameters which we have been working with so far, and, secondly, that we will now *en passant* obtain expressions for the traded volume, which otherwise would have had to be calculated separately.

$$L_{tt'} = \langle \hat{q}(t)\hat{q}(t')\rangle_\star, \quad m_t = \langle \mathrm{sgn}[q(t)]\rangle_\star, \quad k_t = \langle \hat{q}(t)\rangle_\star, \tag{9.110}$$

$$\hat{C}_{tt'} = \frac{i\partial\Phi}{\partial C_{tt'}}, \quad \hat{K}_{tt'} = \frac{i\partial\Phi}{\partial K_{tt'}}, \quad \hat{L}_{tt'} = \frac{i\partial\Phi}{\partial L_{tt'}}, \tag{9.111}$$

$$\hat{m}_t = \frac{i\partial\Phi}{\partial m_t}, \quad \hat{k}_t = \frac{i\partial\Phi}{\partial k_t}. \tag{9.112}$$

The above expressions involve the effective measure $\langle \ldots \rangle_\star$, which is here found to have the form

$$\langle g[\{q,\hat{q}\}]\rangle_\star = \lim_{N\to\infty} \frac{1}{N}\sum_i \left\{ \frac{\int \mathcal{D}q\mathcal{D}\hat{q}\ M_i[\{q,\hat{q}\}]g[\{q,\hat{q}\}]}{\int \mathcal{D}q\mathcal{D}\hat{q}\ M_i[\{q,\hat{q}\}]} \right\}, \tag{9.113}$$

$$\begin{aligned} M_i[\{q,\hat{q}\}] &= p_0(q(0))\ \mathrm{e}^{-i\sum_{tt'}\left[\hat{C}_{tt'}\mathrm{sgn}[q(t)]\mathrm{sgn}[q(t')]+\hat{K}_{tt'}\mathrm{sgn}[q(t)]\hat{q}(t')\right]} \\ &\quad \times \mathrm{e}^{-\mathrm{i}\sum_{tt'}\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')-i\sum_t\left[\hat{m}_t\mathrm{sgn}[q(t)]+\hat{k}_t\hat{q}(t)\right]} \\ &\quad \times \mathrm{e}^{\mathrm{i}\sum_t\left[\hat{q}(t)[q(t+1)-q(t)-\theta_i(t)+\varepsilon_i]+\psi_i(t)\mathrm{sgn}[q(t)]\right]}. \end{aligned} \tag{9.114}$$

9.3.3 The saddle-point equations

In order to proceed towards the elimination of the external fields, we now identify our observables by taking the usual derivatives of the generating functional with respect to $\{\psi_i(t), \theta_i(t)\}$. The resulting expressions involving C and G are identical to those found in our previous models. However, here we have additional one-time observables, which can be interpreted as follows (with the shorthand $\{\mathcal{D}\} = \mathcal{D}C\mathcal{D}\hat{C}\mathcal{D}K\mathcal{D}\hat{K}\mathcal{D}L\mathcal{D}\hat{L}\mathcal{D}m\mathcal{D}\hat{m}$):

$$\begin{aligned} 0 &= \lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N}\sum_i \frac{\partial}{\partial\theta_i(t)}\overline{Z[\boldsymbol{\psi}]} \\ &= \lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{1}{N}\sum_i \frac{\int\{\mathcal{D}\}(\partial/\partial\theta_i(t))\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\ &= \lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}} \sum_i \frac{\int\{\mathcal{D}\}(\partial\Phi/\partial\theta_i(t))\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \\ &= -i\lim_{\boldsymbol{\psi}\to\mathbf{0}}\langle\hat{q}(t)\rangle_\star, \end{aligned} \tag{9.115}$$

$$\begin{aligned} m(t) &= \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle \mathrm{sgn}[q_i(t)]\rangle} = -i\lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}}\frac{1}{N}\sum_i\frac{\partial}{\partial\psi_i(t)}\overline{Z[\boldsymbol{\psi}]} \\ &= -\mathrm{i}\lim_{N\to\infty}\lim_{\boldsymbol{\psi}\to\mathbf{0}}\frac{1}{N}\sum_i \frac{\int\{\mathcal{D}\}(\partial/\partial\psi_i(t))\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}} \end{aligned}$$

$$= -\mathrm{i} \lim_{N\to\infty} \lim_{\boldsymbol{\psi}\to\mathbf{0}} \sum_i \frac{\int\{\mathcal{D}\}(\partial\Phi/\partial\psi_i(t))\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}{\int\{\mathcal{D}\}\mathrm{e}^{N[\Psi+\Phi+\Omega]+\mathcal{O}(N^0)}}$$
$$= \lim_{\boldsymbol{\psi}\to\mathbf{0}} \langle \mathrm{sgn}[q(t)]\rangle_\star. \tag{9.116}$$

It follows that at the physical saddle-point the order parameters $C_{tt'}$ are indeed the entries of the correlation function (9.98), and that at the saddle-point we must have

$$L_{tt'} = 0, \quad K_{tt'} = iG_{tt'}, \quad m_t = m(t), \quad k_t = 0. \tag{9.117}$$

At first sight it appears that the calculation of the function Φ (9.107) is to be done again in full detail, due to the various extra terms that have appeared in the exponential. However, close inspection reveals that if we play our cards right we may save ourselves further integrations: the present function Φ (9.107) can be obtained from (4.43) by making in the latter the substitutions

$$C_{tt'} \to C_{tt'} + m_t + m_{t'}, \quad G_{tt'} \to G_{tt'} - ik_{t'}.$$

We may therefore simply make the relevant substitutions in the result (4.44) of the previous Gaussian integrations, thus redefining the matrix D to

$$D_{tt'} = 1 + C_{tt'} + m_t + m_{t'} \tag{9.118}$$

and expand this formula further for small values of $\{k_t\}$. Upon putting temporarily $Z_{tt'} = -ik_{t'}$ and expanding $\mathrm{Tr}\log(\mathbf{I}+G+Z) = \log\det(\mathbf{I}+G+Z) = \mathrm{Tr}\log(\mathbf{I}+G) + \mathrm{Tr}\log[\mathbf{I}+(\mathbf{I}+G)^{-1}Z] = \mathrm{Tr}\log(\mathbf{I}+G) + \mathrm{Tr}[(\mathbf{I}+G)^{-1}Z] + \mathcal{O}(k^2)$ (based on the matrix identity $\log\det U = \mathrm{Tr}\log U$), this results in

$$\Phi = -\alpha\mathrm{Tr}\log(\mathbf{I}+G+Z) - \frac{1}{2}\alpha\sum_{tt'} L_{tt'}[(\mathbf{I}+G)^\dagger D^{-1}(\mathbf{I}+G)]^{-1}_{tt'} + \mathcal{O}(L^2, kL)$$
$$= -\alpha\mathrm{Tr}\log(\mathbf{I}+G) - \frac{1}{2}\alpha\sum_{tt'} L_{tt'}[(\mathbf{I}+G)^\dagger D^{-1}(\mathbf{I}+G)]^{-1}_{tt'}$$
$$+ i\alpha\sum_{tt'}(\mathbf{I}+G)^{-1}_{tt'}k_t + \mathcal{O}(k^2, kL, L^2). \tag{9.119}$$

We are now in a position to extract those saddle-point equations that involve differentiation of Φ, i.e. the sets (9.111) and (9.112), using $K = iG$, the general identity $\partial\mathrm{Tr}\log U/\partial U_{tt'} = (U^{-1})_{t't}$ and the fact that $\{k, L\} \to 0$ at the physical saddle-point

$$\hat{C}_{tt'} = \lim_{k,L\to 0} \frac{i\partial\Phi}{\partial C_{tt'}} = 0, \tag{9.120}$$

$$\hat{K}_{tt'} = \lim_{k,L\to 0} \frac{\partial \Phi}{\partial G_{tt'}} = -\alpha \frac{\partial}{\partial G_{tt'}} \mathrm{Tr}\log(\mathbb{1}+G) = -\alpha(\mathbb{1}+G^{\dagger})^{-1}_{tt'}, \quad (9.121)$$

$$\hat{L}_{tt'} = \lim_{k,L\to 0} \frac{i\partial \Phi}{\partial L_{tt'}} = -\frac{1}{2} i\alpha[(\mathbb{1}+G)^{\dagger} D^{-1}(\mathbb{1}+G)]^{-1}_{tt'}, \quad (9.122)$$

$$\hat{m}_t = \lim_{k,L\to 0} \frac{i\partial \Phi}{\partial m_t} = 0, \quad (9.123)$$

$$\hat{k}_t = \lim_{k,L\to 0} \frac{i\partial \Phi}{\partial k_t} = -\alpha \sum_{t'} (\mathbb{1}+G)^{-1}_{tt'}. \quad (9.124)$$

We may now put $\boldsymbol{\psi} \to \mathbf{0}$ and choose site-independent external fields $\theta_i(t) = \theta(t)$. The vanishing of $\{\hat{C}, \hat{m}\}$, together with the other expressions above for the conjugate order parameters, simplifies our effective measure (9.114) (which now depends on the site index i solely via the control variable ε_i) to

$$M_i[\{q,\hat{q}\}] = p_0(q(0))\, \mathrm{e}^{\mathrm{i}\sum_t \hat{q}(t)\left[q(t+1)-q(t)-\theta(t)+\varepsilon_i+\alpha\sum_{t'}(\mathbb{1}+G)^{-1}_{tt'}[\mathrm{sgn}[q(t')]+1]\right]} \times \mathrm{e}^{-\mathrm{i}\sum_{tt'}\hat{q}(t)\hat{L}_{tt'}\hat{q}(t')}. \quad (9.125)$$

One subsequently shows in the by now standard manner, via causality, that $\int \mathcal{D}q\mathcal{D}\hat{q}\, M_i[\{q,\hat{q}\}] = 1$. This in turn validates the identity $\langle \mathrm{sgn}[q(t)]\hat{q}(t')\rangle_\star = i\partial\langle \mathrm{sgn}[q(t)]\rangle_\star / \partial\theta(t')$, and we no longer encounter the variables $\{\hat{q}(t)\}$ in our order parameter equations, so that they can be integrated out. We then find our final (exact) dynamical theory for our present MG in the limit $N \to \infty$, in the form of the following closed equations for the dynamic order parameters $\{C, G, m\}$:

$$C_{tt'} = \langle \mathrm{sgn}[q(t)]\mathrm{sgn}[q(t')]\rangle_\star, \quad G_{tt'} = \frac{\partial}{\partial\theta(t')}\langle \mathrm{sgn}[q(t)]\rangle_\star, \quad m_t = \langle \mathrm{sgn}[q(t)]\rangle_\star \quad (9.126)$$

with the following effective measure

$$\langle g[\{q,\hat{q}\}]\rangle_\star = \int \mathrm{d}\varepsilon\, W(\varepsilon) \int \Big[\prod_t \mathrm{d}q(t)\Big] M[\{q\}|\varepsilon] g[\{q\}], \quad (9.127)$$

$$M[\{q\}|\varepsilon] = p_0(q(0)) \int \prod_t \left[\frac{\mathrm{d}\eta(t)}{\sqrt{2\pi}}\right] \frac{\mathrm{e}^{-(1/2)\sum_{tt'}\eta(t)[(\mathbb{1}+G)^{-1}D(\mathbb{1}+G^{\dagger})^{-1}]_{tt'}\eta(t')}}{\sqrt{\det[(\mathbb{1}+G)^{-1}D(\mathbb{1}+G^{\dagger})^{-1}]}} \times \prod_{t\geq 0} \delta\Big[q(t+1)-q(t)-\theta(t)+\varepsilon+2\alpha\sum_{t'}(\mathbb{1}+G)^{-1}_{tt'}\theta[q(t')]-\sqrt{\alpha}\eta(t)\Big] \quad (9.128)$$

and with

$$W(\varepsilon) = \lim_{N\to\infty} \frac{1}{N} \sum_i \delta[\varepsilon - \varepsilon_i]. \tag{9.129}$$

We see that the measure (9.128), which generates the averages in (9.126) for the present model and which is parametrized by the values of the parameter ε, corresponds to the following discrete-time effective single agent process

$$q(t+1) = q(t) + \theta(t) - \varepsilon - 2\alpha \sum_{t'\leq t} (\mathbf{1}+G)^{-1}_{tt'} \theta[q(t')] + \sqrt{\alpha}\,\eta(t) \tag{9.130}$$

in which $\eta(t)$ is a Gaussian noise, with zero mean and with temporal correlations given by $\langle \eta(t)\eta(t')\rangle = \Sigma_{tt'}$:

$$\Sigma = (\mathbf{1}+G)^{-1} D (\mathbf{1}+G^{\dagger})^{-1}. \tag{9.131}$$

However, in contrast to our previous models, here the entries of the kernel D are given by $D_{tt'} = 1 + C_{tt'} + m_t + m_{t'}$. Similar to our previous MG (the version with trend followers), one now has to solve first the effective single agent equation (9.130) for a given choice of ε, then calculate covariances, averages, and response functions corresponding to this value of ε, and finally average the latter three quantities over the distribution $W(\varepsilon)$.

9.3.4 Overall bid statistics and volatility

Given the established identification of the overall bids $A(t)$ in the MG with the returns in the market that it is supposed to model, we will obviously be interested in both the volatility σ and the full bid distribution $P(A)$. We thus return to the bid generating function $Z[\boldsymbol{\phi}]$ (4.57), which is here obtained from the generating function $Z[\boldsymbol{\psi}]$ (9.100) by the substitution

$$\mathrm{e}^{\mathrm{i}\sum_{it}\psi_i(t)\mathrm{sgn}[q_i(t)]} \;\to\; \mathrm{e}^{\mathrm{i}\sqrt{2}\sum_{\mu t}\phi_\mu(t)A_\mu(t)} = \mathrm{e}^{\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{i\mu t}\phi_\mu(t)R_i^\mu\theta[q_i(t)]}.$$

Comparison with the first line of (9.101) shows that, after a suitable shift in the integration variables w_t^μ, we can obtain $\overline{Z[\boldsymbol{\phi}]}$ from the last line of (9.101) by simply putting $\boldsymbol{\psi}\to\mathbf{0}$ and replacing $i\sum_{t\mu}\hat{w}_t^\mu w_t^\mu \to i\sum_{t\mu}\hat{w}_t^\mu[w_t^\mu - \phi_\mu(t)/\sqrt{2}]$, so

$$\overline{Z[\boldsymbol{\phi}]} = \int \mathcal{D}\mathbf{q}\mathcal{D}\widehat{\mathbf{q}}\; p_0(\mathbf{q}(0))\; \mathrm{e}^{\mathrm{i}\sum_{it}\hat{q}_i(t)[q_i(t+1)-q_i(t)-\theta_i(t)+\varepsilon_i]} \tag{9.132}$$
$$\times \int \mathcal{D}\mathbf{w}\mathcal{D}\widehat{\mathbf{w}}\; \mathrm{e}^{\mathrm{i}\sum_{t\mu}\hat{w}_t^\mu[w_t^\mu-\frac{1}{\sqrt{2}}\phi_\mu(t)]-\frac{1}{2N}\sum_{i\mu}\left(\sum_t\left[2w_t^\mu\theta[q_i(t)]-\hat{w}_t^\mu\hat{q}_i(t)\right]\right)^2+\mathcal{O}(N^0)}.$$

From here we may proceed along the established road towards a saddle-point integral, which gives us again a formula for $\overline{Z[\boldsymbol{\phi}]}$ in the standard form (9.105), with the same expressions for the functions Ψ (9.106) and Ω (9.108) (provided we put $\boldsymbol{\psi} \to \mathbf{0}$), but with the function Φ being replaced by

$$\Phi = \frac{1}{N}\sum_{\mu} \log\left[\int \mathcal{D}w\mathcal{D}\hat{w}\ \mathrm{e}^{-(1/2)\sum_{tt'}[w_t w_{t'}(1+m_t+m_{t'}+C_{tt'})+\hat{w}_t\hat{w}_{t'}L_{tt'}]} \right.$$
$$\left. \times\ \mathrm{e}^{\sum_{tt'} w_t\hat{w}_{t'}(k_{t'}+i\delta_{tt'}+K_{tt'})-\frac{i}{\sqrt{2}}\sum_t \phi_\mu(t)\hat{w}_t}\right]. \tag{9.133}$$

Having already derived our saddle-point equations, we may now put $\{k, L\} \to 0$ (which we know to be the case at the physical saddle-point) as well as $K = iG$ and $1+m_t+m_{t'}+C_{tt'} = D_{tt'}$, and thereby find our expression (9.133) for Φ simplifying to

$$\Phi = \frac{1}{N}\sum_{\mu} \log \int \mathcal{D}w\mathcal{D}\hat{w}\ \mathrm{e}^{\mathrm{i}\sum_t \hat{w}_t\left[\sum_{t'}(\mathbf{I}+G^\dagger)_{tt'}w_{t'}-(\phi_\mu(t)/\sqrt{2})\right]-(1/2)\sum_{tt'} w_t D_{tt'} w_{t'}}. \tag{9.134}$$

We see that this expression can, in turn, be obtained from the first line of (4.65) by making in the latter the replacement $\phi_\mu(t) \to \phi_\mu(t)/\sqrt{2}$. Thus we can immediately proceed to the last line of (4.65) and make the same replacement there, which gives us (using the identity $\det(\mathbf{I}+G^\dagger) = 1$ at the physical saddle-point):

$$\Phi = -\frac{1}{4N}\sum_{\mu}\sum_{tt'} \phi_\mu(t)[(\mathbf{I}+G)^{-1}D(\mathbf{I}+G^\dagger)^{-1}]_{tt'}\phi_\mu(t'). \tag{9.135}$$

From this it follows, firstly, upon using $\overline{Z[\boldsymbol{\phi}]}$ as a generating function exactly as in Chapter 4, that the disorder-average bid covariance matrix in the present model is given by

$$\begin{aligned}
\overline{\Xi}_{tt'} &= \lim_{N\to\infty} \frac{1}{p}\sum_{\mu} \overline{\langle A^\mu[\mathbf{q}(t)]A^\mu[\mathbf{q}(t')]\rangle} \\
&= -\lim_{N\to\infty}\lim_{\boldsymbol{\phi}\to\mathbf{0}} \frac{1}{2p}\sum_{\mu} \frac{\partial^2 \overline{Z[\boldsymbol{\phi}]}}{\partial\phi_\mu(t)\partial\phi_\mu(t')} \\
&= \frac{1}{4}[(\mathbf{I}+G)^{-1}D(\mathbf{I}+G^\dagger)^{-1}]_{tt'},
\end{aligned} \tag{9.136}$$

where D and G take their values at the physical saddle-point.

With the result (9.135), and given that our present definition of the generating function $\overline{Z[\boldsymbol{\phi}]}$ is identical to that employed in Chapter 6, we are are now also in a position to repeat the calculation of the instantaneous and long-time bid distributions $P_t(A)$ and $P(A)$ in Chapter 6, and obtain

$$\begin{aligned}
P_t(A) &= \int \frac{\mathrm{d}\hat{A}}{2\pi}\, \mathrm{e}^{-\mathrm{i}\hat{A}A} \lim_{N\to\infty} \frac{1}{p} \sum_{\mu=1}^{p} \overline{Z[\hat{A}\hat{\mathbf{e}}^{\mu}(t)/\sqrt{2}]} \\
&= \int \frac{\mathrm{d}\hat{A}}{2\pi}\, \mathrm{e}^{-\mathrm{i}\hat{A}A} \lim_{N\to\infty} \frac{\int\{\mathcal{D}\}\mathrm{e}^{N[\Psi+\Phi+\Omega]-(1/8)\hat{A}^2[(\mathbf{1}+G)^{-1}D(\mathbf{1}+G^{\dagger})^{-1}]_{tt}}}{\int\{\mathcal{D}\}\mathrm{e}^{N[\Psi+\Phi+\Omega]}} \\
&= \int \frac{\mathrm{d}\hat{A}}{2\pi}\, \mathrm{e}^{-\mathrm{i}\hat{A}A-(1/8)\hat{A}^2[(\mathbf{1}+G)^{-1}D(\mathbf{1}+G^{\dagger})^{-1}]_{tt}} = \mathrm{e}^{-(1/2)\hat{A}^2\Xi_{tt}}.
\end{aligned} \tag{9.137}$$

We conclude that the formulae (6.6)–(6.8) of the standard batch MG again apply, provided we now insert the new expression (9.136) for the bid covariance matrix

$$P_t(A) = \int \frac{\mathrm{d}\hat{A}}{2\pi}\, \mathrm{e}^{-\mathrm{i}\hat{A}A-(1/2)\hat{A}^2\overline{\Xi}_{tt}} = \frac{\mathrm{e}^{-(1/2)A^2/\overline{\Xi}_{tt}}}{\sqrt{2\pi\overline{\Xi}_{tt}}}, \tag{9.138}$$

$$P(A) = \int \frac{\mathrm{d}\Xi\, W(\Xi)}{\sqrt{2\pi\Xi}} \mathrm{e}^{-(1/2)A^2/\Xi}, \tag{9.139}$$

$$W(\Xi) = \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=1}^{\tau} \delta[\Xi - \overline{\Xi}_{tt}]. \tag{9.140}$$

Thus, even in the present MG model where our agents are able to move in or out of the market on the basis of their perceived likelihood of having profitable transactions, with batch dynamics non-Gaussian bid statistics can occur at most when the system operates exactly at criticality, i.e. for parameter values that put the MG exactly at a phase transition (where one could find that the slow critical relaxation of $\overline{\Xi}_{tt}$ prevents (9.140) from being a δ-function[67]).

9.3.5 TTI stationary states

We now return to the effective single-agent equation (9.130) and solve it along the established lines via the combined *ansätze* of TTI and absence of anomalous response, i.e. $C_{tt'} = C(t-t')$, $G_{tt'} = G(t-t')$, $m_t = m$, and $\chi = \sum_t G(t) < \infty$. We also define

[67] However, even at criticality it is not realistic to expect ‘fat tails’ in the distribution $P(A)$ of *any* batch version of the MG. The observable $\overline{\Xi}_{tt}$ will continue to evolve (albeit slowly at the critical point) from the finite ‘random trading’ value $\overline{\Xi}_{00}$ (here equal to $\frac{1}{2}$, and equal to 1 in our previous MGs) to the stationary value σ^2. This implies that, even if it is not a δ-peak, $W(\Xi)$ will still not have support for large values of Ξ as long as our batch MG models have a finite volatility at the critical point, which implies that in expression (9.139) the tails of $P(A)$ will still be Gaussian.

the persistent correlations $c = \lim_{t\to\pm\infty} C(t)$. For a given value of the parameter ε we write (9.130) in integrated form, and introduce $\tilde{q}(t) = q(t)/t$:

$$\tilde{q}(t) = \frac{q(0)}{t} + \frac{1}{t}\sum_{t'=0}^{t-1}\left[\theta(t') - \varepsilon + \sqrt{\alpha}\,\eta(t')\right] - \frac{2\alpha}{t}\sum_{t'=0}^{t-1}\sum_{s}(\mathbb{1}+G)^{-1}_{t',s}\theta[q(s)].$$

We send $t \to \infty$, write time averages generally as $\overline{x} = \lim_{t\to\infty} t^{-1}\sum_{t'<t} x(t')$, and find the following equation for $\tilde{q} = \lim_{t\to\infty}\tilde{q}(t)$:

$$\tilde{q} = \sqrt{\alpha}\overline{\eta} + \overline{\theta} - \varepsilon - \frac{2\alpha\overline{\theta[q]}}{1+\chi}. \tag{9.141}$$

Before solving this equation and calculating for any given value of ε the fundamental quantity $\overline{\sigma} = \overline{\mathrm{sgn}[q]} = 2\overline{\theta[q]} - 1$, let us briefly note that

$$\begin{aligned}\chi &= \frac{\partial}{\partial\overline{\theta}}\int \mathrm{d}\varepsilon\, W(\varepsilon)\int \mathrm{d}\overline{\eta}P(\overline{\eta})\overline{\sigma}\Big|_{\varepsilon} \\ &= -\int \mathrm{d}\varepsilon\, W(\varepsilon)\frac{\partial}{\partial\varepsilon}\int \mathrm{d}\overline{\eta}P(\overline{\eta})\overline{\sigma}\Big|_{\varepsilon}.\end{aligned} \tag{9.142}$$

This relation, as always, enables us to put $\overline{\theta} \to 0$ (the external fields have served their purpose), and simplify (9.141) to

$$\tilde{q} = \sqrt{\alpha}\overline{\eta} - \varepsilon - \frac{2\alpha\overline{\theta[q]}}{1+\chi}. \tag{9.143}$$

This equation can be solved without difficulty. In contrast to the previous model with trend followers, there is no positive feedback and hence the conditions of existence for the possible solutions are once more fully complementary, so that the value of $\overline{\sigma} = \overline{\mathrm{sgn}[q]}$ can be written directly and exactly as a function of $\overline{\eta}$. However, what will be different here is that the signs of $\overline{\eta}$ and $\tilde{q}$ become more important, as they will control the tendency of agents to participate in the game. We find the solutions of (9.143) to be the following

$$\overline{\eta} < \frac{\varepsilon}{\sqrt{\alpha}}: \quad \text{inactive solution,} \quad \overline{\sigma} = -1, \tag{9.144}$$

$$\frac{\varepsilon}{\sqrt{\alpha}} < \overline{\eta} < \frac{\varepsilon}{\sqrt{\alpha}} + \frac{2\sqrt{\alpha}}{1+\chi}: \quad \text{fickle solution,} \quad \overline{\sigma} = \frac{1+\chi}{\sqrt{\alpha}}\left[\overline{\eta} - \frac{\varepsilon}{\sqrt{\alpha}}\right] - 1, \tag{9.145}$$

$$\overline{\eta} > \frac{\varepsilon}{\sqrt{\alpha}} + \frac{2\sqrt{\alpha}}{1+\chi}: \quad \text{active solution,} \quad \overline{\sigma} = 1. \tag{9.146}$$

The solution with $\overline{\sigma} = -1$ describes the sub-population of agents who have decided to withdraw from the market; their trading strategies are found to be of insufficient quality to tempt them into playing. The solution with $\overline{\sigma} = 1$ describes those agents who will *always* submit a bid, since on average they will benefit sufficiently (the criterion for 'sufficiently' being set by the parameter ε). Finally, the fickle solution represents the group of agents who will trade now and then, never becoming full-time players but never withdrawing from the market either; they play only at times when they believe their trading strategy will generate profit. We see that for $\varepsilon \to -\infty$, not playing will no longer be an option, whereas for $\varepsilon \to \infty$ our agents will just leave the market; this is clearly consistent with our initial interpretation of the role assigned to the parameter ε.

We now only need to know the variance of the zero-average persistent Gaussian noise $\overline{\eta}$ in order to proceed to the calculation of closed equations for our persistent order parameters. From (9.131) and the definition (9.118) it follows that this variance is given by

$$\begin{aligned}\langle \overline{\eta}^2 \rangle &= \lim_{\tau\to\infty} \tau^{-2} \sum_{tt'\le\tau} \Sigma(t-t') = \Sigma(\infty) \\ &= [(\mathbf{I}+G)^{-1} D (\mathbf{I}+G^\dagger)^{-1}](\infty) = \frac{1+2m+c}{(1+\chi)^2}.\end{aligned} \tag{9.147}$$

It will be found advantageous to define the following two shorthands

$$v = \frac{\sqrt{2\alpha}}{\sqrt{1+2m+c}}, \quad w = \frac{\varepsilon(1+\chi)}{\sqrt{2\alpha(1+2m+c)}}. \tag{9.148}$$

We may now write $\overline{\eta} = z\sqrt{2\alpha}/v(1+\chi)$, where z is a zero-average and unit-variance Gaussian variable, and the relations between $\overline{\sigma}$ and $\overline{\eta}$ for the three solution types translate into

$$\frac{z}{\sqrt{2}} < w: \quad \text{inactive solution}, \quad \overline{\sigma} = -1, \tag{9.149}$$

$$w < \frac{z}{\sqrt{2}} < w+v: \quad \text{fickle solution}, \quad \overline{\sigma} = \frac{\sqrt{2}(z - w\sqrt{2})}{v} - 1, \tag{9.150}$$

$$\frac{z}{\sqrt{2}} > w+v: \quad \text{active solution}, \quad \overline{\sigma} = 1. \tag{9.151}$$

Rather than having just one quantity ϕ to measure the fraction of frozen agents, it seems sensible in the present model to distinguish between the permanently inactive and the permanently active frozen agents, so we will here define two such quantities $\phi_\pm$, giving the fractions of trajectories for the effective agent with $\overline{\sigma} = \pm 1$, respectively. The persistent order parameters can now be written in a form similar to that introduced for the previous model, as integrals over contributions from the sub-populations characterized by their value of ε (all these integrals are of a type that we have already run into in earlier chapters):

$$c = \int \mathrm{d}\varepsilon\, W(\varepsilon)\, c(\varepsilon), \quad \phi_{\pm} = \int \mathrm{d}\varepsilon\, W(\varepsilon)\, \phi_{\pm}(\varepsilon), \tag{9.152}$$

$$m = \int \mathrm{d}\varepsilon\, W(\varepsilon)\, m(\varepsilon), \quad \chi = -\int \mathrm{d}\varepsilon\, W(\varepsilon)\, \frac{\partial}{\partial\varepsilon} m(\varepsilon), \tag{9.153}$$

in which we have the following ε-parametrized persistent observables

$$\begin{aligned}
c(\varepsilon) &= \int Dz\, \overline{\sigma}^2[z] = 1 + \int_{w\sqrt{2}}^{(w+v)\sqrt{2}} Dz\, \left\{ \Big[\frac{\sqrt{2}(z - w\sqrt{2})}{v} - 1\Big]^2 - 1 \right\} \\
&= 1 + \frac{2}{v^2} \int_{w\sqrt{2}}^{(w+v)\sqrt{2}} Dz\, \Big\{ (z - w\sqrt{2})^2 - \sqrt{2}v(z - w\sqrt{2}) \Big\} \\
&= 1 + \frac{2}{v^2} \int_{w\sqrt{2}}^{(w+v)\sqrt{2}} Dz\, z^2 - \frac{2\sqrt{2}(2w+v)}{v^2} \int_{w\sqrt{2}}^{(w+v)\sqrt{2}} Dz\, z \\
&\quad + \frac{4w(w+v)}{v^2} \int_{w\sqrt{2}}^{(w+v)\sqrt{2}} Dz \\
&= 1 + \frac{1 + 2w(w+v)}{v^2} \{\mathrm{Erf}[w+v] - \mathrm{Erf}[w]\} \\
&\quad + \frac{2w}{v^2\sqrt{\pi}} \mathrm{e}^{-(w+v)^2} - \frac{2(w+v)}{v^2\sqrt{\pi}} \mathrm{e}^{-w^2},
\end{aligned} \tag{9.154}$$

$$\begin{aligned}
m(\varepsilon) &= \int_{(w+v)\sqrt{2}}^{\infty} Dz - \int_{-\infty}^{w\sqrt{2}} Dz + \int_{w\sqrt{2}}^{(w+v)\sqrt{2}} Dz\, \left\{ \frac{\sqrt{2}(z - w\sqrt{2})}{v} - 1 \right\} \\
&= \frac{\sqrt{2}}{v} \int_{w\sqrt{2}}^{(w+v)\sqrt{2}} Dz\, (z - w\sqrt{2}) - \mathrm{Erf}[w+v] \\
&= \frac{1}{v\sqrt{\pi}} \Big[\mathrm{e}^{-w^2} - \mathrm{e}^{-(w+v)^2}\Big] - \Big(\frac{w}{v} + 1\Big) \mathrm{Erf}[w+v] + \frac{w}{v} \mathrm{Erf}[w],
\end{aligned} \tag{9.155}$$

$$\phi_{+}(\varepsilon) = \int_{(w+v)\sqrt{2}}^{\infty} Dz = \frac{1}{2} - \frac{1}{2}\mathrm{Erf}[w+v], \tag{9.156}$$

$$\phi_{-}(\varepsilon) = \int_{-\infty}^{w\sqrt{2}} Dz = \frac{1}{2} + \frac{1}{2}\mathrm{Erf}[w]. \tag{9.157}$$

Insertion into (9.152) and (9.153) gives us a closed theory describing the ergodic states of our MG in terms of persistent order parameters only. To avoid confusion we now write $w = \varepsilon u$ (note that v and u do not depend on ε). One then finds

$$c = 1 + \frac{1}{v^2} \int \mathrm{d}\varepsilon\, W(\varepsilon)\, [1 + 2\varepsilon u(\varepsilon u + v)] \Big\{ \mathrm{Erf}[\varepsilon u + v] - \mathrm{Erf}[\varepsilon u] \Big\}$$

$$+\frac{1}{v^2\sqrt{\pi}}\int \mathrm{d}\varepsilon\, W(\varepsilon)\left\{2\varepsilon u \mathrm{e}^{-(\varepsilon u+v)^2} - 2(\varepsilon u + v)\mathrm{e}^{-\varepsilon^2 u^2}\right\}, \quad (9.158)$$

$$m = \frac{1}{v\sqrt{\pi}}\int \mathrm{d}\varepsilon\, W(\varepsilon)\left[\mathrm{e}^{-\varepsilon^2 u^2} - \mathrm{e}^{-(\varepsilon u+v)^2}\right],$$

$$-\frac{1}{v}\int d\varepsilon\, W(\varepsilon)\Big\{(\varepsilon u + v)\mathrm{Erf}[\varepsilon u + v] - \varepsilon u \mathrm{Erf}[\varepsilon u]\Big\}, \quad (9.159)$$

$$\phi_+ = \frac{1}{2} - \frac{1}{2}\int d\varepsilon\, W(\varepsilon)\ \mathrm{Erf}[\varepsilon u + v], \quad (9.160)$$

$$\phi_- = \frac{1}{2} + \frac{1}{2}\int d\varepsilon\, W(\varepsilon)\ \mathrm{Erf}[\varepsilon u], \quad (9.161)$$

$$\alpha\chi = (1+\chi)(1 - \phi_+ - \phi_-). \quad (9.162)$$

In these equations one can regard $\{u, v, \phi_\pm\}$ simply as shorthands, leaving a closed theory in terms of $\{c, m, \chi\}$ only.

9.3.6 Predictability and approximated volatility

Although different from our previous models, the quantity $\overline{\Xi}(\infty)$, which measures the predictability of the overall bids in the market (see Chapter 6), can again be expressed without approximation in terms of our persistent observables. In TTI stationary states we may use $m_t = m$, $D_{tt'} = 1 + 2m + C(t - t')$, and $\sum_t (\mathbf{I} + G)^{-1} = (1 + \chi)^{-1}$. This allows us to take the appropriate limit in (9.136) and write

$$\overline{\Xi}(\infty) = \frac{1}{4}[(\mathbf{I} + G)^{-1} D (\mathbf{I} + G^\dagger)^{-1}](\infty) = \frac{1 + 2m + c}{4(1+\chi)^2}. \quad (9.163)$$

Also the earlier derivations of the approximations for the volatility in terms of persistent order parameters only (in TTI stationary states), i.e. (4.154) and (4.155) for the standard batch MG, will be different in the present model. Rather than redoing these derivations in full, let us here concentrate only on (4.155), which in the past has proven to be the more accurate approximation of the two. The approximation is, as before, based on separating the correlation function in a contribution from frozen trajectories and one from fickle trajectories, followed by assuming that fickle trajectories do not experience a retarded self-interaction. Our starting point is the volatility matrix (9.136), of which the diagonal elements in the stationary state reduce to the volatility

$$\sigma^2 = \overline{\Xi}(0) = \frac{1}{4}[(\mathbf{I} + G)^{-1} D (\mathbf{I} + G^\dagger)^{-1}](0)$$

$$= \frac{1 + 2m}{4(1+\chi)^2} + \frac{1}{4}\sum_{ss'} (\mathbf{I} + G)^{-1}(s)\langle \mathrm{sgn}[q(s)]\mathrm{sgn}[q(s')]\rangle_\star (\mathbf{I} + G)^{-1}](s')$$

$$
\begin{aligned}
&= \frac{1+2m}{4(1+\chi)^2} + \frac{\phi_+ + \phi_-}{4(1+\chi)^2} + \frac{1}{4}\int \mathrm{d}\varepsilon\, W(\varepsilon)[1-\phi_+(\varepsilon)-\phi_-(\varepsilon)] \\
&\quad \times \Big\langle \Big[\sum_s (\mathbb{1}+G)^{-1}(s)\mathrm{sgn}[q(s)] \Big]^2 \Big\rangle|_{\mathrm{fickle},\varepsilon} \\
&= \frac{1+2m}{4(1+\chi)^2} + \frac{\phi_+ + \phi_-}{4(1+\chi)^2} + \frac{1}{4}\int \mathrm{d}\varepsilon\, W(\varepsilon) W(\mathrm{fickle}|\varepsilon) \\
&\quad \times \Big\langle \Big[2\sum_s (\mathbb{1}+G)^{-1}(s)\theta[q(s)] - \frac{1}{1+\chi} \Big]^2 \Big\rangle|_{\mathrm{fickle},\varepsilon}.
\end{aligned} \tag{9.164}
$$

We observe that the sum $\sum_s \cdots$ in the last line is exactly the retarded self-interaction term in the effective single-agent equation (9.130). Our approximation consists, as before, in assuming that for fickle agents only the instantaneous term (i.e. $s = 0$) in this sum contributes. We may then put $2\sum_s \cdots \to 2\theta[q(0)] = \mathrm{sgn}[q(0)] + 1$, which leads us to

$$
\begin{aligned}
\sigma^2 &= \frac{1+2m}{4(1+\chi)^2} + \frac{\phi_+ + \phi_-}{4(1+\chi)^2} + \frac{1}{4}\int \mathrm{d}\varepsilon\, W(\varepsilon) W(\mathrm{fickle}|\varepsilon) \Big\langle \Big[\mathrm{sgn}[q(0)] + \frac{\chi}{1+\chi} \Big]^2 \Big\rangle|_{\mathrm{fickle},\varepsilon} \\
&= \frac{1+2m}{4(1+\chi)^2} + \frac{\phi_+ + \phi_-}{4(1+\chi)^2} + \frac{1}{4}\int \mathrm{d}\varepsilon\, W(\varepsilon) W(\mathrm{fickle}|\varepsilon) \\
&\quad \times \left\{ 1 + \frac{\chi^2}{(1+\chi)^2} + \frac{2\chi}{1+\chi} \langle \mathrm{sgn}[q(0)] \rangle \big|_{\mathrm{fickle},\varepsilon} \right\} \\
&= \frac{1+2m}{4(1+\chi)^2} + \frac{\phi_+ + \phi_-}{4(1+\chi)^2} + \frac{1}{4}(1-\phi_+ - \phi_-)\Big[1 + \frac{\chi^2}{(1+\chi)^2} \Big] \\
&\quad + \frac{\chi}{2(1+\chi)} \int \mathrm{d}\varepsilon\, W(\varepsilon) W(\mathrm{fickle}|\varepsilon) \langle \mathrm{sgn}[q(0)] \rangle \big|_{\mathrm{fickle},\varepsilon}.
\end{aligned} \tag{9.165}
$$

We may now work out the last line with help of the following (exact) relation

$$
\begin{aligned}
m &= \int \mathrm{d}\varepsilon\, W(\varepsilon) m(\varepsilon) = \int \mathrm{d}\varepsilon\, W(\varepsilon) \langle \mathrm{sgn}[q(0)] \rangle |_{\varepsilon} \\
&= \int \mathrm{d}\varepsilon\, W(\varepsilon) \left\{ W(\mathrm{fickle}|\varepsilon) \langle \mathrm{sgn}[q(0)] \rangle |_{\mathrm{fickle},\varepsilon} + W(\mathrm{frozen}|\varepsilon) \langle \mathrm{sgn}[q(0)] \rangle |_{\mathrm{frozen},\varepsilon} \right\} \\
&= \int \mathrm{d}\varepsilon\, W(\varepsilon) W(\mathrm{fickle}|\varepsilon) \langle \mathrm{sgn}[q(0)] \rangle |_{\mathrm{fickle},\varepsilon} + \int \mathrm{d}\varepsilon\, W(\varepsilon) \{ \phi_+(\varepsilon) - \phi_-(\varepsilon) \} \\
&= \int \mathrm{d}\varepsilon\, W(\varepsilon) W(\mathrm{fickle}|\varepsilon) \langle \mathrm{sgn}[q(0)] \rangle |_{\mathrm{fickle},\varepsilon} + \phi_+ - \phi_-.
\end{aligned} \tag{9.166}
$$

Combination with (9.165), followed by some cosmetic rearranging of terms, then gives us the desired approximation for the volatility in terms of persistent order parameters, which takes a surprisingly simple form

$$
\sigma^2 = \frac{1+m}{2(1+\chi)^2} + \frac{\chi(1+m-2\phi_+)}{2(1+\chi)}. \tag{9.167}
$$

9.3.7 Strict producers and uniform speculators—theory

At this stage one has to make a choice for the function $W(\varepsilon)$, which controls the tendencies and temperaments of the agents in the game, with $\varepsilon > 0$ implying reluctance to place a bid (unless the agent is confident of success) and $\varepsilon < 0$ implying the opposite. The simplest choice that preserves the notion of having produces and speculators is a bi-modal one, with a fraction p of producers who must *always* trade (i.e. they have $\varepsilon = -\infty$) and a fraction $1-p$ of speculators with a uniform level of caution (i.e. they have $\varepsilon_i = \varepsilon^\star$, for some $\varepsilon^\star \in \mathbb{R}$):

$$W(\varepsilon) = p\delta(\varepsilon + \infty) + (1-p)\delta(\varepsilon - \varepsilon^\star). \tag{9.168}$$

Insertion into our order parameter equations (9.158)–(9.162), which involves sending $\varepsilon \to -\infty$ for the producers, simplifies these equations to

$$\begin{aligned} c = 1 &+ \frac{1-p}{v^2}\,[1 + 2\varepsilon^\star u(\varepsilon^\star u + v)]\Big\{\mathrm{Erf}[\varepsilon^\star u + v] - \mathrm{Erf}[\varepsilon^\star u]\Big\} \\ &+ \frac{1-p}{v^2\sqrt{\pi}}\left\{2\varepsilon^\star u \mathrm{e}^{-(\varepsilon^\star u + v)^2} - 2(\varepsilon^\star u + v)\mathrm{e}^{-(\varepsilon^\star u)^2}\right\}, \end{aligned} \tag{9.169}$$

$$\begin{aligned} m = p &+ \frac{1-p}{v\sqrt{\pi}}\left[\mathrm{e}^{-(\varepsilon^\star u)^2} - \mathrm{e}^{-(\varepsilon^\star u + v)^2}\right] \\ &- \frac{1-p}{v}\Big\{(\varepsilon^\star u + v)\mathrm{Erf}[\varepsilon^\star u + v] - \varepsilon^\star u \mathrm{Erf}[\varepsilon^\star u]\Big\}, \end{aligned} \tag{9.170}$$

$$\phi_+ = \frac{1}{2}(1+p) - \frac{1}{2}(1-p)\mathrm{Erf}[\varepsilon^\star u + v], \tag{9.171}$$

$$\phi_- = \frac{1}{2}(1-p) + \frac{1}{2}(1-p)\mathrm{Erf}[\varepsilon^\star u], \tag{9.172}$$

$$\alpha\chi = (1+\chi)(1 - \phi_+ - \phi_-), \tag{9.173}$$

with the two shorthands

$$v = \frac{\sqrt{2\alpha}}{\sqrt{1 + 2m + c}}, \quad u = \frac{1+\chi}{\sqrt{2\alpha(1 + 2m + c)}}. \tag{9.174}$$

From the solution of these coupled equations, which will generally have to be found numerically, then follows the volatility (in approximation) via (9.167).

However, it is instructive to work out the above equations in special limits, where we can anticipate how the system should behave:

Riscophobic speculators: $\varepsilon^\star \to \infty$

In this limit we must expect all speculators to have $q_i < 0$, so that the system should be in a state where all producers trade permanently and all speculators have left the market. Our order parameter equations confirm this picture, as they are seen to reduce to

$$c = 1, \quad m = 2p - 1, \quad \chi = 0, \quad \phi_+ = p, \quad \phi_- = 1 - p.$$

Riscophilic speculators: $\varepsilon^\star \to -\infty$

In this limit we must expect all speculators to have $q_i > 0$, and we should find a market consisting effectively of producers only. Again this is borne out by our order parameter equations, which here simplify to

$$c = m = 1, \quad \chi = 0, \quad \phi_+ = 1, \quad \phi_- = 0.$$

Pure producers' markets: $p \to 1$

In this limit we should retrieve exactly the previous solution of riscophilic speculators. This indeed turns out to be the case, as it should be.

Pure speculators' markets: $p \to 0$

Here we must expect the problem to remain non-trivial, as our speculating agents can still be in any of the three relevant groups (always playing, never playing, and fickle), depending on their appetite for risk as controlled by $\varepsilon^\star$. Indeed one finds

$$\begin{aligned}
c &= 1 + \frac{1}{v^2}\,[1 + 2\varepsilon^\star u(\varepsilon^\star u + v)]\Big\{\mathrm{Erf}[\varepsilon^\star u + v] - \mathrm{Erf}[\varepsilon^\star u]\Big\} \\
&\quad + \frac{1}{v^2\sqrt{\pi}}\left\{2\varepsilon^\star u \mathrm{e}^{-(\varepsilon^\star u + v)^2} - 2(\varepsilon^\star u + v)\mathrm{e}^{-(\varepsilon^\star u)^2}\right\}, \\
m &= \frac{1}{v\sqrt{\pi}}\left[\mathrm{e}^{-(\varepsilon^\star u)^2} - \mathrm{e}^{-(\varepsilon^\star u + v)^2}\right] \\
&\quad - \frac{1}{v}\Big\{(\varepsilon^\star u + v)\mathrm{Erf}[\varepsilon^\star u + v] - \varepsilon^\star u \mathrm{Erf}[\varepsilon^\star u]\Big\}, \\
v &= \frac{\sqrt{2\alpha}}{\sqrt{1 + 2m + c}}, \qquad u = \frac{1 + \chi}{\sqrt{2\alpha(1 + 2m + c)}}, \\
\phi_+ &= \frac{1}{2} - \frac{1}{2}\mathrm{Erf}[\varepsilon^\star u + v], \qquad \phi_- = \frac{1}{2} + \frac{1}{2}\mathrm{Erf}[\varepsilon^\star u], \\
\alpha\chi &= (1 + \chi)(1 - \phi_+ - \phi_-).
\end{aligned}$$

Let us next try to construct a phase diagram, by locating the occurrence of a $\chi \to \infty$ phase transition (if any), marking the breakdown of ergodicity and the boundary of the regime of validity of our above equations. To do so, we first rewrite (9.173) by using $u/v = (1 + \chi)/2\alpha$, in order to make all occurrences of χ explicit, giving

$$\frac{\chi}{1 + \chi} = \frac{1 - p}{2\alpha}\left\{\mathrm{Erf}\left[\frac{\varepsilon^\star v(1 + \chi)}{2\alpha} + v\right] - \mathrm{Erf}\left[\frac{\varepsilon^\star v(1 + \chi)}{2\alpha}\right]\right\}. \qquad (9.175)$$

Since $|m| \leq 1$ and $c \in [0, 1]$ we may infer from the definition (9.174) that always $v \geq \frac{1}{2}\sqrt{2\alpha}$. This implies that, unless $\varepsilon^\star = 0$, and apart from the pathological and/or trivial cases $\alpha = 0$ and $p = 1$ the right-hand side of (9.175) must always go to zero whenever $\chi \to \infty$, whereas the left-hand side will go to one. We thus arrive at the somewhat surprising conclusion that there cannot be a $\chi \to \infty$ transition for any $\varepsilon^\star \neq 0$. This does not imply, however, that there is no transition at all: in contrast, we will see that for $\varepsilon^\star \neq 0$ and sufficiently small p (i.e. a sufficiently large fraction of speculators) there will instead be a *discontinuous* transition to a low α regime with multiple solutions.

For $\varepsilon^\star = 0$, in contrast, a $\chi \to \infty$ transition can occur. Here our persistent order parameter equations are seen to simplify dramatically. If we interpret the resulting formulae for c and m as functions of v, i.e.

$$c(v) = 1 + \frac{1-p}{v^2}\left\{\mathrm{Erf}[v] - \frac{2v}{\sqrt{\pi}}\right\}, \tag{9.176}$$

$$m(v) = p + (1-p)\left\{\frac{1}{v\sqrt{\pi}}[1-\mathrm{e}^{-v^2}] - \mathrm{Erf}[v]\right\} \tag{9.177}$$

we find, for any choice of the remaining control parameters $\{\alpha, p\}$, the solution(s) of our order parameter equations to be expressed explicitly in terms of the fixed-point(s) $v^\star$ of the following non-linear map

$$F(v) = \frac{\sqrt{2\alpha}}{\sqrt{1+2m(v)+c(v)}}. \tag{9.178}$$

The corresponding values of c and m are subsequently obtained by insertion of $v^\star$ into (9.176) and (9.177), whereas the remaining observables $\{\chi, \phi_\pm\}$ follow from

$$\chi = \frac{(1-p)\mathrm{Erf}[v]}{2\alpha - (1-p)\mathrm{Erf}[v]}, \tag{9.179}$$

$$\phi_- = \frac{1}{2}(1-p), \qquad \phi_+ = \frac{1}{2}(1+p) - \frac{1}{2}(1-p)\mathrm{Erf}[v]. \tag{9.180}$$

A $\chi \to \infty$ phase transition is seen to occur when $\alpha = \frac{1}{2}(1-p)\mathrm{Erf}[v]$. This further condition, combination with the fixed-point condition for the mapping (9.178), gives the phase transition line in the (α, p) plane. After some minor re-arrangements and substitutions one finds that this line can be written in a convenient parametrized form $(\alpha_c(v), p_c(v))$, where $v \geq 0$ and with

$$\alpha(v) = \frac{\mathrm{Erf}[v]}{\mathrm{Erf}[v] + (\mathrm{e}^{-v^2}/v\sqrt{\pi}) + 1}, \quad p(v) = \frac{\mathrm{Erf}[v] + (\mathrm{e}^{-v^2}/v\sqrt{\pi}) - 1}{\mathrm{Erf}[v] + (\mathrm{e}^{-v^2}/v\sqrt{\pi}) + 1}. \tag{9.181}$$

This transition line separates an ergodic large α regime where the overall bids are partly predictable (since $\overline{\Xi}(\infty) > 0$) from a non-ergodic small α regime where the

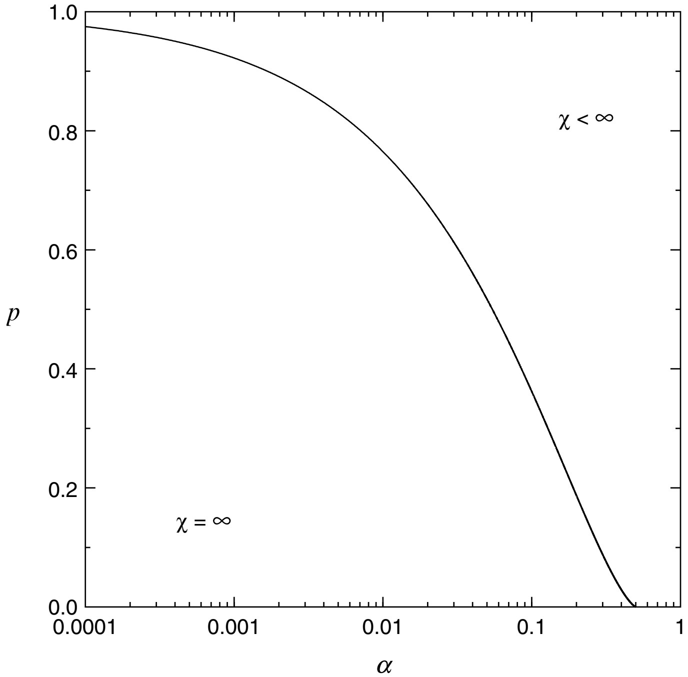


Fig. 9.9 Phase diagram of the (batch dynamics and random history) MG with a fraction p of producers and a fraction $1-p$ of speculators, consisting of an ergodic high α phase with TTI and $\chi < \infty$ (where the overall bids are partly predictable) and a non-ergodic low α phase where $\chi = \infty$ (where the bids are not predictable). The producers trade at each time step (they all have $\varepsilon_i = -\infty$), whereas the speculators trade only when they believe they will make profit (here with $\varepsilon_i = \varepsilon^\star = 0$).

overall bids are no longer predictable[68] (since $\overline{\Xi}(\infty) = 0$). Upon checking the limits $v \to 0$ and $v \to \infty$ in the parametrization (9.181) we see that in a market with producers only one must find $\alpha_c = 0$, whereas in a market with only speculators (and where these speculators all have $\varepsilon_i = 0$) one must find $\alpha_c = \frac{1}{2}$. The result is shown in Fig. 9.9.

Solving the $\varepsilon^\star = 0$ order parameter equations (9.176)–(9.178) numerically for different combinations of the control parameters (α, p), followed by calculation of the remaining persistent observables, leads to graphs such as those in Fig. 9.10. Figures 9.9 and 9.10 together capture and illustrate more or less the main features of the phenomenology of the present model for $\varepsilon^\star = 0$:

[68] Provided we assume, as we have so far done successfully in our previous MG versions, that after having diverged at the phase transition, the susceptibility χ will remain infinite all the way down to $\alpha = 0$.

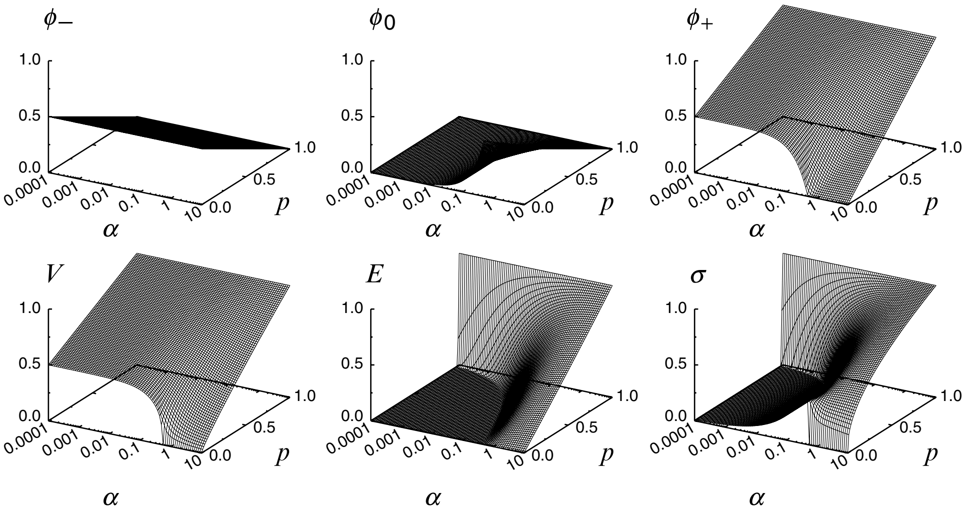


Fig. 9.10 TTI stationary state solution of the (batch dynamics and random history) MG with a fraction p of producers and a fraction $1 - p$ of speculators. The producers trade at each time step (they all have $\varepsilon_i = -\infty$), whereas the speculators trade only when they believe they will make profit (here with $\varepsilon_i = \varepsilon^\star = 0$. The observables shown, from top left to bottom right, are: ϕ_- (the fraction of agents who retired), $\phi_0 = 1 - \phi_+ - \phi_-$ (the fraction of 'fickle' agents), ϕ_+ (the fraction of agents who trade permanently), $V = \frac{1}{2}(1 + m)$ (the traded volume, i.e. $N^{-1} \sum_i |b_i|$), $E = \overline{\Xi}(\infty)$ (the measure of market predictability), and σ (the volatility).

- One observes in the phase diagram that increasing the fraction of speculators in the market, i.e. lowering p, shifts the critical point to larger values of α. Since α is inversely proportional to the number of agents in the market, this implies that now a *smaller* number of agents will be required to eliminate all predictability from the market. In contrast to producers, speculators are able to reduce the amount of exploitable information in the MG.
- Granted the approximations that were necessary to arrive at formula (9.167), in Fig. 9.10 the volatility is seen to decrease monotonically upon increasing the fraction of speculators (i.e. upon lowering p).
- Speculators can only make profit if either there are also producers in the market, or (in the case where there are no producers, so $p = 0$) when the number of speculators is sufficiently large (i.e. $\alpha < \alpha_c = \frac{1}{2}$).

The bottom line is that, at least for $\varepsilon^\star = 0$, the introduction of speculators into an MG market of producers makes this market more efficient, both from the point of view of fluctuations and in terms of reducing market predictability.

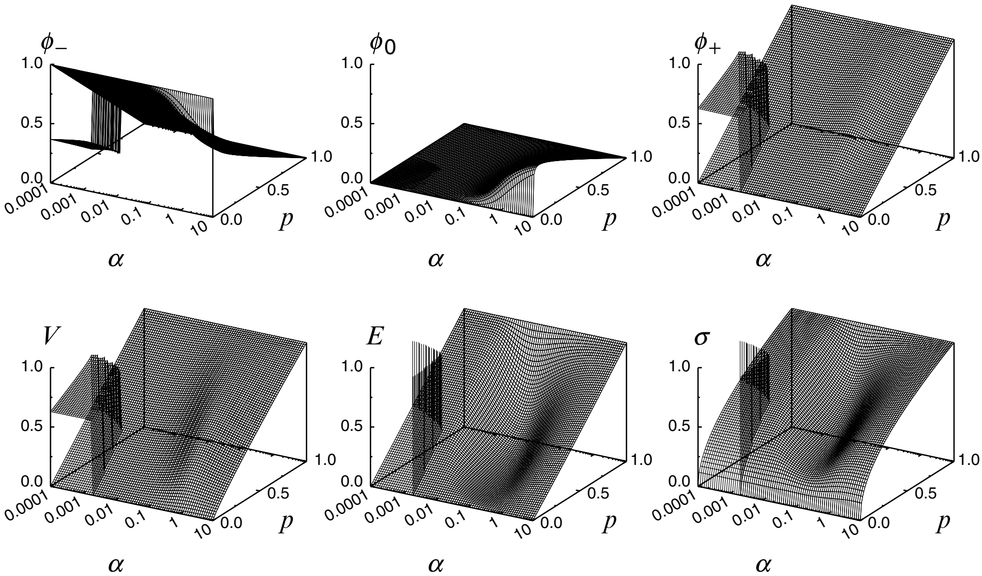


Fig. 9.11 TTI stationary state solution of the (batch dynamics and random history) MG with a fraction p of producers and a fraction $1-p$ of speculators. The producers trade at each time step (they all have $\varepsilon_i=-\infty$), whereas the speculators trade only when they believe they will make profit (here with $\varepsilon_i=\varepsilon^\star=0.25$, so the speculators are somewhat cautious). The observables shown, from top left to bottom right, are: ϕ_- (the fraction of agents who retired), $\phi_0=1-\phi_+-\phi_-$ (the fraction of 'fickle' agents), ϕ_+ (the fraction of agents who trade permanently), $V=\frac{1}{2}(1+m)$ (the traded volume, i.e. $N^{-1}\sum_i|b_i|$), $E=\Xi(\infty)$ (the measure of market predictability), and σ (the volatility).

If one carries out the same numerical exercise for $\varepsilon^\star\neq 0$, i.e. if one solves numerically the more complicated set of equations (9.169), (9.170), (9.173) and (9.174), one finds graphs such as those in Figs. 9.11 and 9.12 (referring to $\varepsilon^\star=0.25$ and $\varepsilon^\star=-0.25$, respectively). These reveal that our coupled equations for persistent observables again exhibit phase transitions (i.e. bifurcations of multiple solutions at sufficiently small values of α), in spite of the susceptibility χ always remaining finite, but that for $\varepsilon^\star\neq 0$ these transitions are of a *discontinuous* nature. Due to the more complicated form and the larger dimensionality of our present equations when compared with the previous case $\varepsilon^\star=0$, it is no longer a trivial matter to determine the precise location of the phase transition line in the (α,p) plane; any numerical search routine written for that purpose will be very sensitive to its initalization. Hence the graphs in Figs 9.11 and 9.12 are to be regarded as reliable in the sense that they do give precise values for the order parameters which solve our equations, and that whenever these graphs show multiple solutions for a given point (α,p) we know that we have crossed the transition

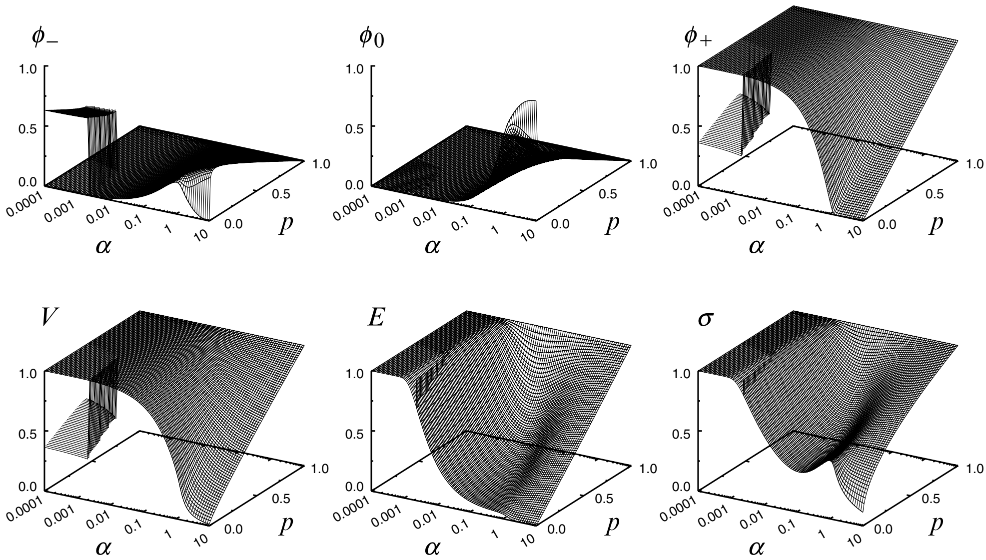


Fig. 9.12 TTI stationary state solution of the (batch dynamics and random history) MG with a fraction p of producers and a fraction $1-p$ of speculators. The producers trade at each time step (they all have $\varepsilon_i = -\infty$), whereas the speculators trade only when they believe they will make profit (here with $\varepsilon_i = \varepsilon^\star = -0.25$, so the speculators are somewhat keen to play). The observables shown, from top left to bottom right, are: ϕ_- (the fraction of agents who retired), $\phi_0 = 1 - \phi_+ - \phi_-$ (the fraction of 'fickle' agents), ϕ_+ (the fraction of agents who trade permanently), $V = \frac{1}{2}(1+m)$ (the traded volume, i.e. $N^{-1}\sum_i |b_i|$), $E = \overline{\Xi}(\infty)$ (the measure of market predictability), and σ (the volatility).

line. However, it might well be that multiple solutions exist already for values of α that are larger than those where they are seen to be picked up by our present numerical routines. Secondly, as soon as we have lowered α below the transition point $\alpha_c(p)$ for a given value of p, one finds that the newly bifurcated solution has $1 + \chi < 0$. This inequality violates the basis of our calculation of the persistent observables, so that (as was the case for the more conventional $\chi = \infty$ transitions) the set of equations (9.169), (9.170), (9.173), and (9.174) no longer applies in the regime $\alpha < \alpha_c(p)$. Hence the discontinuously bifurcating solutions found and shown in Figs 9.11 and 9.12 for small α and p are only ad hoc continuations of the correct $\alpha > \alpha_c(p)$ theory. Upon comparing the data in Figs. 9.11 and 9.12 with the corresponding $\varepsilon^\star = 0$ surfaces as in Fig. 9.10, we may conclude:

- Also for $\varepsilon^\star \neq 0$ it is generally true that the introduction of speculators into a market of producers will improve the efficiency of this market, in reducing both the volatility and the predictability.

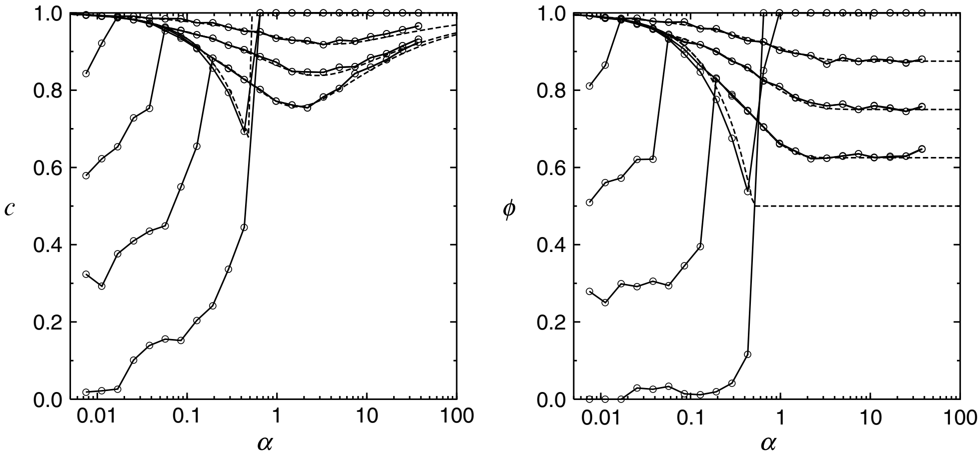


Fig. 9.13 The predicted persistent correlations c (dashed lines, left) and the fraction of frozen agents ϕ (dashed lines, right), together with simulation data (connected markers), for the batch MG with 'fake' histories and composed of a fraction p of producers together with a fraction $1-p$ of speculators. The producers trade at each time step (they all have $\varepsilon_i = -\infty$), whereas the speculators trade only when they believe they will make profit (here with $\varepsilon_i = \varepsilon^\star = 0$). Results are shown for $p \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, from bottom to top. Initial conditions: $|q_i(0)| = \Delta \in \{0, 10\}$ for all i.

- For $\varepsilon^\star > 0$ (risk-averse speculators) the speculators cause a reduction of the trading volume V, whereas for $\varepsilon^\star < 0$ (speculators keen to trade) they cause an increase in the trading volume.
- The efficiency of the market depends non-monotonically on the parameter $\varepsilon^\star$ (which measures the attitude of the speculators to risk); the most efficient market is obtained for $\varepsilon^\star = 0$ (see also Fig. 9.16 below).

9.3.8 Strict producers and uniform speculators—comparison with simulations

Again, numerical simulations confirm the above theoretical results perfectly[69] in their regime of validity (i.e. for $\alpha > \alpha_c(p)$). See, for instance, Figs 9.13–9.16. One notes in practice (at least with the standard types of initial conditions and measurement times) that only in the case where the phase transition is continuous, i.e. for $\varepsilon^\star = 0$, will the system be tempted to go into the high volatility state for sufficiently small values of α. This is clearly seen in Figs 9.13 and 9.14. For non-zero values of $\varepsilon^\star$ in contrast (see the data for $\varepsilon^\star = \pm 0.1$ in Figs 9.15 and 9.16) one tends to find the system in the low volatility branch, irrespective of the choice of initial conditions. The only

[69] The only caveat is that the present model is more prone to fluctuations for large values of α, where the number N of agents relative to the number $p = 2^M$ of fake histories is smallest.

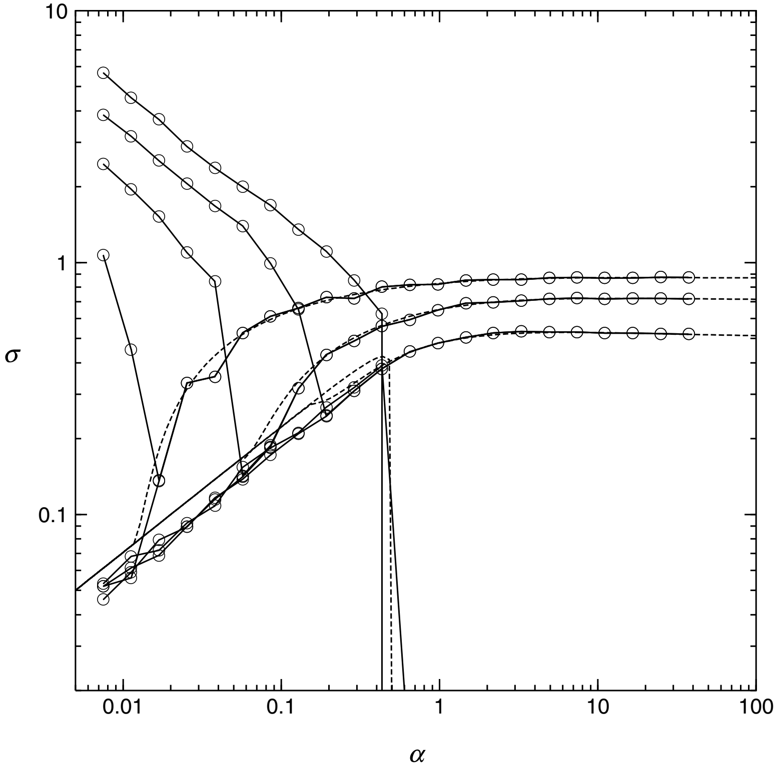


Fig. 9.14 The predicted volatility σ (dashed lines), together with simulation data (connected markers), for the batch MG with 'fake' histories and composed of a fraction p of producers together with a fraction $1-p$ of speculators, as in the previous figure. Results are shown for $p \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$, from bottom to top. Initial conditions: $|q_i(0)| = \Delta \in \{0, 10\}$ for all i.

serious exception to the general agreement between theory and experiment appears to be the behaviour of the system for large α as seen for the choice $p = 0$ (i.e. an MG with speculators only) and $\varepsilon^\star = 0$. Here the simulation data suggest that $\phi = 1$ for $\alpha > \alpha_c$, contradicting the theory. The explanation for this deviation is a subtle one, as for $p = \varepsilon^\star = 0$ it turns that that we have a somewhat degenerate case. Here our persistent order parameter equations reduce to

$$c = 1 + \frac{1}{v^2}\left[\mathrm{Erf}[v] - \frac{2v}{\sqrt{\pi}}\right], \quad m = \frac{1}{v\sqrt{\pi}}\left[1 - \mathrm{e}^{-v^2}\right] - \mathrm{Erf}[v], \tag{9.182}$$

$$v^2 = \frac{2\alpha}{1+2m+c}, \quad \phi = 1 - \frac{1}{2}\mathrm{Erf}[v], \quad 1+\chi = \frac{2\alpha}{2\alpha - \mathrm{Erf}[v]}. \tag{9.183}$$

Substitution of the first two expressions, for c and m respectively, into the equation for v leads to an equation for v only, of the form $\alpha = F(v)$, where

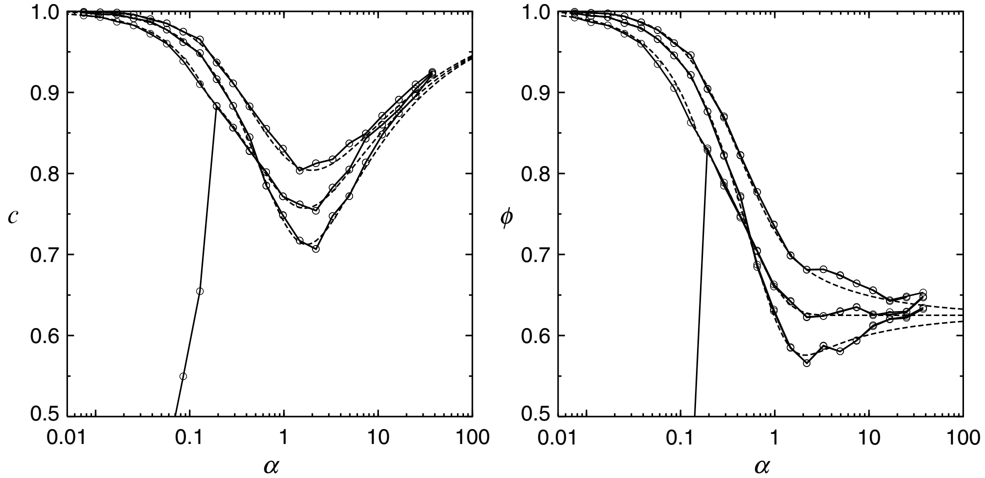


Fig. 9.15 The predicted persistent correlations c (dashed lines, left) and the fraction of frozen agents ϕ (dashed lines, right), together with simulation data (connected markers), for the batch MG with 'fake' histories and composed of a fraction p of producers with a fraction $1-p$ of speculators. The producers trade at each time step (they all have $\varepsilon_i = -\infty$), whereas the speculators trade only when they believe they will make profit (here with $\varepsilon_i = \varepsilon^\star$). Results are shown for $p = 0.25$ and $\varepsilon^\star \in \{-0.1, 0, 0.1\}$, from bottom to top for large α. Initial conditions: $|q_i(0)| = \Delta \in \{0, 10\}$ for all i.

$$F(v) = v^2 - \frac{v}{\sqrt{\pi}}\mathrm{e}^{-v^2} + \left(\frac{1}{2} - v^2\right)\mathrm{Erf}[v] \qquad (9.184)$$

The function F is seen to obey

$$F(v) = v^2 + \mathcal{O}(v^3), \quad F'(v) = 2v(1 - \mathrm{Erf}[v]), \quad F(\infty) = \frac{1}{2}.$$

The equation $\alpha = F(v)$ will therefore only have a solution if $\alpha < \frac{1}{2}$, but this solution is not acceptable as it has $1+\chi < 0$. At $\alpha = \frac{1}{2}$ one finds the degenerate frozen solution $v = \infty, c = 1, m = -1, \phi = \frac{1}{2}$, and with χ being ill-defined. This particular solution, however, reveals the origin of the apparent deviation between theory and experiment. For $m = -1$ one has $V = N^{-1}\sum_i \overline{\langle |b_i| \rangle} = 0$, which suggests that the system must be frozen into a state with $\phi_- = 1$, and hence also $\phi = 1$. Yet the same solution claims that $\phi = \frac{1}{2}$. The explanation for this apparent inconsistency is that in the degenerate case $p = \varepsilon^\star = 0$ one has agents with diverging negative values of $q_i(t)$ (hence they will never trade), but where these values do not diverge linearly so that they are still being classified as fickle. This mechanism can be seen at work in equation (9.150), which (since $w = 0$ as a result of $\varepsilon^\star = 0$) shows that for $v \to \infty$ we must have half of the agents in a state that is formally classified as fickle (i.e. the valuation differences

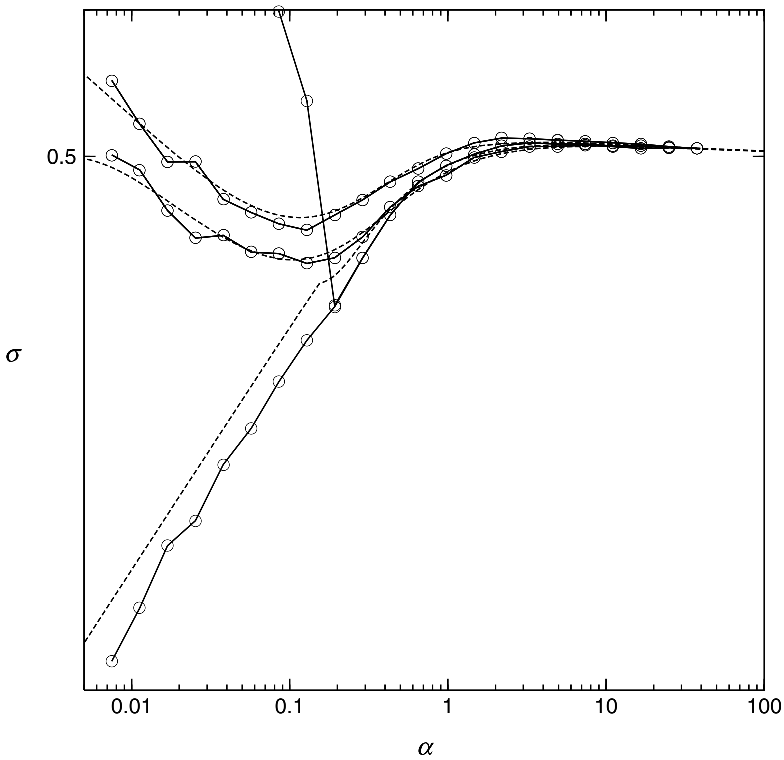


Fig. 9.16 The predicted volatility σ (dashed lines), together with simulation data (connected markers), for the batch MG with 'fake' histories and composed of a fraction p of producers together with a fraction $1-p$ of speculators, as in the preview figure. Results are shown for $p = 0.25$ and $\varepsilon^\star \in \{-0.1, 0, 0.1\}$ (low α region: lowest value for $\varepsilon^\star = 0$, followed by $\varepsilon = 0.1$, with the highest value for $\varepsilon^\star = -0.1$). Initial conditions: $|q_i(0)| = \Delta \in \{0, 10\}$ for all i.

do not grow linearly with time), but where still $\overline{\sigma} = -1$ (i.e. the valuation differences are permanently negative).

9.3.9 Economic relevance and realism

Models where agents have the option of not playing if they believe that playing would not be profitable (which are sometimes called 'grand canonical' MGs) have traditionally been those versions of the MG for which observations of stylized facts in numerical simulations have been reported in literature. These included fat tails of the overall bid and volume distributions and volatility clustering. Careful reading of especially the earlier such reports, however, reveals that: (i) stylized facts were usually reported for rather small system sizes (mostly $N < 500$), and (ii) only at the critical point of the

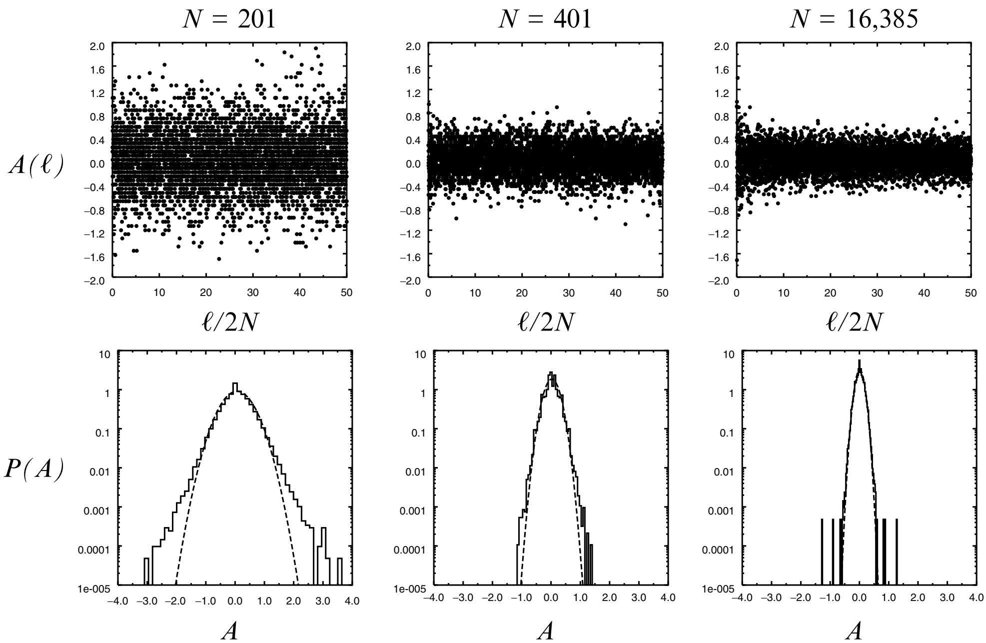


Fig. 9.17 Top: numerical simulation measurements of the re-scaled total bids $A(\ell)$ in the on-line MG with producers and speculators, for different system sizes N but all operating at criticality: $(\alpha, p) = (0.1, 0.362)$ and $\varepsilon^\star = 0$ (so that the transition can indeed be located precisely). Bottom: the corresponding distributions $P(A)$ of these observed values (represented as histograms, and plotted logarithmically), together with Gaussian distributions (dashed) of width and average identical to those of the observed $P(A)$.

model at hand (so that finding these effects also required tuning of control parameters), (iii) evidence for volatility clustering has been of a somewhat anecdotal character (rather than appearing on a regular basis). In some papers one can even deduce that volatility clustering was observed in parameter regimes where the distribution $P(A)$ did not have fat tails. Only very recently have such phenomena been subjected to a simple finite size analysis, measuring their strength for different system sizes. This led to the conclusion that fat tails of the bid (i.e. price) distribution $P(A)$ and of the volume distribution are in fact finite size effects. They all disappear when we increase the system size[70], see Fig. 9.17 (for the on-line version of the present model, since the batch version as always exhibits Gaussian bid distributions only). Comparison with e.g. Figs 9.1 and 9.8 also shows, however, that finite size fluctuations in the present model are much more prominent than they were in the previous MG versions.

[70] Plotting figures similar to Fig. 9.17 for the distribution of trading volumes $V = \frac{1}{2}(1 + m)$ in the present model gives very narrowly peaked distributions, without fat tails, not even for small system sizes.

We conclude that, from an economic point of view, MG models with producers and speculators show interesting and complex new modes of behaviour, when compared with their predecessors, such as discontinuous phase transitions and much more prominent finite size fluctuations. The introduction of speculators generally improves the efficiency of the market[71]. It is also clear from the solution of our macroscopic equations that for $p > 0$ (i.e. in the presence of producers) there exists a clear optimum value for the parameter $\varepsilon^\star$ which controls the appetite for risk of the speculators. This optimal value, within the present definitions given by $\varepsilon^\star_{\rm opt} = 0$, minimizes simultaneously the volatility and the predictability of the market. As a result of the stronger finite size fluctuations in the present model with producers and speculators it is here also much easier to observe some of the stylized facts of financial markets. However, these are all finite size phenomena, which will have disappeared effectively already for system sizes around $N = 1,000$, which furthermore require tuning of control parameters in order to be observed, and which by their very nature are not described by any of the existing mathematical theories (since the latter all involve the limit $N \to \infty$).

[71] Only for negative values of $\varepsilon^\star$ (risk loving speculators) and small α (i.e. a crowded market) does the introduction of speculators have only a negligible effect.

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10. Notes

10.1 Strengths and weaknesses of MG theory

Let us accept the rough but not unreasonable distinction of the development of most new mathematics-based research fields into the following three stages: a first stage where a simple but original new model is being proposed with which to explain a phenomenon that is not yet understood (or perhaps a range of phenomena), a second stage where the most appropriate mathematical techniques are being selected, adapted, or developed from scratch (as the case may be) with which to tackle this new model, and a third stage where there is general agreement on the mathematical tools to be used and where the emphasis consequently has shifted to the application of these accepted tools to increasingly refined and realistic versions of the initial model. It would seem that in the field of MGs we are now approaching the third stage. The question is no longer *how* to solve MG type models, but rather *which* version or generalization of the MG to solve, from the infinite set of possibilities.

10.1.1 The theorist's perspective

In the MG we have been fortunate that the relevant mathematical techniques were already available (they have been developed and refined in the field of the statistical mechanics of disordered and complex systems, from the 1970s onwards), and required only modest adaptation. At present one will be able to calculate phase diagrams and TTI stationary solutions for most generalizations of the MG with fake market histories, following by now established procedures. Such results will be exact in the infinite system size limit, and lead to predictions for persistent observables such as the fraction of frozen agents, the persistent correlations, the susceptibility, and the measure of market predictability. There is no problem in having agent diversity. However, apart from simply keeping up with the new variations of the MG that continue to be proposed, many interesting theoretical problems are still open. Of these, some are of a fundamental nature (possibly requiring the development of new mathematical methods or tricks), whereas others should be quite easily solved with presently available methods but are just waiting to be given attention. To name but a few:

- The volatility cannot be expressed in terms of persistent order parameters, but within the generating functional analysis we have an exact expression which involves also the short-time parts of the correlation and response functions $\{C, G\}$. One would like to find exact solutions of the dynamic order parameter equations for short times, so that we can go beyond using only approximations for the volatility.
- The statics and dynamics of the MG in the non-ergodic regime (i.e. for small α) have never been analyzed properly. Although this regime is described by exact closed equations, these equations have so far only been solved using crude approximations (see Chapter 4).
- One would like to understand in much more detail the bid distribution $P(A)$ of the on-line MG versions in the non-ergodic regime. This includes the origin and scaling properties of the outliers in $P(A)$, which seem to move outward to larger values of $|A|$ as the history depth M is being increased. It also includes calculating the non-Gaussian tails of the central part of $P(A)$. Again, exact formulae for $P(A)$ are available, but for on-line models these involve non-trivial Poissonnian averages (describing the fluctuations induced by the randomly drawn pseudo-histories).
- It has been possible to solve the on-line MG for the case of having real market histories, but this solution takes the form of an expansion in the width of the distribution of history string frequencies. This implies that in its present form it can be used only in models where the impact of having real histories is modest. It would therefore be very welcome if an alternative solution method could be developed which would also apply to cases where the history string frequency distribution is far from a δ-peak.
- Within the domain of real versus fake history another interesting question is what would be the effect of having *consistent* fake memory, which can be seen as an intermediate choice in between the two cases of real and (inconsistently) fake memory that have been studied so far.

10.1.2 The economist's perspective

The advantage to the economist of having a solvable model such as the MG is that it allows for the rigorous[72] testing of proposals regarding the origin and statistical

[72] One should be aware that in MG literature the word 'exact' appears to have been subject to some inflation.

characteristics of fluctuations in markets. An exact solution not only replaces an infinite number of simulation experiments, it also allows for precise extrapolation and prediction. Thus, agent-based models such as the MG can not only be used as a tool with which to understand the properties of financial time series in the real world, and the impact of having agent species such as trend followers and speculators on the complex macroscopic processes in markets, but also to hint at what such financial time series could have been if the microscopic rules of the system were to be changed. The most striking example of this is perhaps the astonishing effect in the MG of agents correcting for their own impact on the market bid (and hence the price), see Chapter 7, which reduces the volatility by orders of magnitude and changes the dynamics of the market completely.

The dominant issue at this point in time in MG research, from an economics point of view, would appear to be to continue the quest for the stylized facts:

- In a nutshell, in the present MG versions one can observe some of the stylized facts of financial time series, but one *has to look for them*. This contrasts sharply with economic reality, where they are more or less impossible to avoid. In on-line versions of the MG one does find non-Gaussian tails in the distribution of prices in the non-ergodic regime, even in the infinite system size limit, but at present they would seem to be exponential rather than algebraic. All the other observations of stylized facts are now accepted to be finite size effects at phase transitions, which require sufficiently small system sizes and tuning of control parameters.
- Irrespective of whether small system sizes are themselves realistic economically, they present us with a fundamental problem. The reason for seeking an explanation for the non-Gaussian statistics of financial time series is that if these had been an aggregate of a large number of independent and more or less similar contributions, Gaussian statistics would have been an inevitable consequence of the central limit theorem. If we now accept small system sizes as an explanation, then there would not have been an issue in the first place: the central limit theorem would simply not have applied.
- The tuning of control parameters required for the MG to be at criticality could be avoided if one takes seriously a proposal in literature that predictable markets, by their promise of profit, draw in speculators, whereas unpredictable markets drive speculators out. This mechanism would automatically keep the market at the phase transition point (provided it is a $\chi = \infty$ transition, where indeed the predictibility vanishes). This idea, however, has not yet been implemented and elaborated mathematically.
- If one rejects the idea of stylized facts being finite size effects, one will have to introduce new mechanisms into the MG. These could take the form of evolving

capital of agents, and variable and capital dependent bid magnitudes (so far such models have been studied only via numerical simulations), or of delay mechanisms in agents' actions (which are always guaranteed to cause further instabilities). Another alternative would be to choose more sophisticated mathematical representations of agents' attitudes to risk, for instance, by allowing agents to be minority seekers in 'normal' circumstances (i.e. for modest values of the overall bid), but to become trend followers in extreme cases of very large overall bid values.

New ingredients such as agents' action delays or non-monotonic response to market observations should not pose fundamental obstacles to the application of generating functional methods. The main problem for the mathematical theorist is no longer how to find methods with which to solve such generalized models, but how to avoid wasting energy by solving models that are of only limited economic relevance. It follows from the above that perhaps the time has come for a greater involvement of economists at the model design stage.

10.2 Bibliographical notes

The emphasis in this book is on the mathematical analysis of MGs. This is reflected both in the selection of material and in the depth of its coverage of mathematical derivations. As such, it can perhaps be regarded as complementing another recent book on MGs [1], which offers alternative perspectives, such as their relation with other types of market models and the interpretation of the stationary states of the MG in game theoretic terms.

The birth of MG in 1997 generated a large number of (mainly simulation) studies, exploring the model and proposing and investigating variations. The appealing simplicity of its definition allows anyone to experiment with the MG on a simple desktop PC, and many have accepted this implicit invitation. It is neither possible nor helpful to the reader to discuss or even list each and every one of the papers that have been published. Instead, apart from the references on mathematical methods, the notes below represent an inevitably subjective selection of MG papers, on the basis of their relevance (as perceived by the author) for the development and mathematical understanding of the model.

10.2.1 Mathematical methods

The general background area of probability theory and stochastic processes provides the relevant probabilistic concepts and language. Standard references are [2] for theory and applications of probability theory and [3, 4] for theory and applications of stochastic processes. In addition, one relies heavily in a text such as the present, which aims for explicit solutions wherever possible, on an enormous body of mathematical results on specific functions, expansions, series, integrals, etc. Many of these can be found in [5, 6] (or, in abbreviated form, in [7]). Various practical methods of a physical nature can be found in [8], whereas the saddle-point method is explained in more detail in [9]. Good introductions to the field of statistical mechanics are [10, 11], whereas the specific use of path integrals in dealing with stochastic processes is described in [12]. More on the temporal regularization of discrete time stochastic processes is found in [13]. For elegant introductions to the role of statistical mechanical techniques in the quantitative analysis of problems in economy one might consult [14, 15].

10.2.2 Definition and phenomenology of the game

The El Farol bar problem, of which the MG is more or less a mathematical implementation, was proposed in [16]. This was followed by the introduction and definition of the MG in [17]. The main subsequent early papers in which the crucial phenomenological aspects of the MG were uncovered (albeit not in the order in which they have been presented in this book) would appear to be [18] (the identification of $\alpha = 2^M/N$ as the main control parameter of the MG, the non-monotonic dependence of the volatility on α), [19] (a detailed simulation study of the overall bid statistics for large system sizes), [20] (the introduction of the concept of 'frozen agents'), and [21] (the surprising similarity between the behaviour of MG models with real or fake market histories).

10.2.3 Replica analysis of MGs

The first serious attempt at solving the MG was based on replica analysis. For more details on replica theory as a formalism for solving models of disordered physical systems, including its mathematical subtleties such as replica symmetry and the breaking of replica symmetry, one could study the classics [22, 23] or the more recent textbook [24] (which deals also with various applications of replica theory in other disciplines). The replica analysis of the MG (with fake market histories) was developed in stages, starting with [25, 26] (deterministic approximation of microscopic laws), and followed by a more advanced version [27] where fluctuations were partly included (via a Fokker–Planck description, with approximated diffusion terms). Further applications of the replica approach include MG models with decay in the valuation dynamics [28] or with replica symmetry breaking [29] (some results of this paper were later revised in [30]).

10.2.4 Generating functional analysis of MGs

The generating functional analysis (GFA) of disordered (physical) systems was launched originally in [31]. The method is also discussed in the textbook [23], again in the context of disordered magnetic materials in physics. The first applications to MGs are found in [32] (the batch MG with fake market histories) and [33] (dealing with the on-line version). These were followed by GFA studies of MGs with market self-impact correction [30] (as introduced earlier in [34] and [29]), of batch MGs with decision noise [35,36] or correlated strategies [37], and of the spherical version of the MG [38].

10.2.5 Theory of MGs with inhomogeneous agent populations

Batch MG versions with inhomogenous decision noise levels were first solved in [35]. Models with trend followers as such seem to go back to [39] (where one had trend followers only, i.e. a Majority Game), but the inhomogeneous version with both trend followers and minority seekers (or 'contrarians') was solved in [40]. A somewhat similar type of model (which page limitations have prevented us from discussing in this book) involves agents trading at different frequencies [41,42] (by having strategy look-up tables with entries $\{-1, 0, 1\}$, with different densities of zeros for different agents).

10.2.6 True versus fake market histories in MGs

After [21] revealed the similarity between the behaviour of MG models with real versus fake market histories, it was shown that this statement did not extend to various variations of the MG, such as games with slightly different strategy valuation update rules [43] or with populations where different agents do not necessarily observe history strings of the same length [44]. A nice (partly phenomenological) attempt at analyzing quantitatively the effects of true history in the MG was presented in [45], and followed by a detailed simulation study [46] of bid periodicities induced by having real histories. The more elaborate analysis of the MG with true market history as described in this book is found in [47].

10.2.7 Producers, speculators, and the hunt for stylized facts

Attempts to reproduce the stylized facts of financial markets have a considerable history. An early (simulation) study where MG agents are allowed the decision not to play (i.e. with speculators *avant la lettre*) is [48]. A number of subsequent papers attempted to solve models with speculators (approximately) using replica methods, such as [34] (here speculators cannot yet leave the game but have two strategies, and producers only one), [49] (with speculators, producers, and also money, similar to [50]), and [51–53] which study more or less the MG version with producers and speculators as discussed in this book (although not yet with generating functional methods).

10.2.8 Alternative choices of models and methods

An alternative definition of the MG involves replacing the traditional conversion of observed external (market) information into a scalar bid (which so far has been implemented via 'look-up tables') by a construction where external information is represented by a vector, and strategies by linear functionals on the space of these external information vectors. Such models, introduced in [54] and elaborated further in [55,56], have played an important role in settling the discussion of to what extent the fluctuations in the bid valuation dynamics of the MG can be neglected, but (somewhat surprisingly) have not yet been solved. This is why they have not been discussed in this book.

Many authors have proposed generalizations of the MG with some form of Darwinism in the market, where, for instance, poorly performing agents are replaced ([57] or [58]), or in which agents can obtain new strategies [59, 60]. So far such models have mainly been studied via numerical simulations.

An alternative approach to the analysis of MGs with fake market histories was proposed under the name of crowd–anticrowd theory, see [61–63]. A recent and detailed exposé is given in [64]. Compared with the other mathematical techniques that are available, the 'crowd–anticrowd' method appears to have serious drawbacks: it relies on several moderately controlled assumptions and approximations, it leads only to predictions for the volatility, and it can in fact not deal with the MG as such, but only with a simplified model where all strategies are either identical, un-correlated or anti-correlated.

10.2.9 Recent directions

Let us finally mention some recent developments of relevance to the mathematical analysis of MGs. The first concerns 'diluted' MGs [65], where each agent can use in his valuation updates only the overal bid of a (frozen) randomly drawn subset of the community. This is found to cause an MO type phase transition (similar to that in MGs with self-impact correction). The second area is that of MGs where agents do not memorize their strategy valuations indefinitely. Such systems, which exhibit novel and interesting fluctuation phenomena, were already studied in [28] (using replica theory) and [66, 67] (via simulations), but are now being studied using generating functional methods [68].

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APPENDIX A: Simple mathematical conventions and tools

A.1 Notation conventions

In this book we will use the following mathematical shorthands

$$\text{Kronecker delta}: \ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases} \tag{A.1}$$

$$\text{Gaussian measure}: \ Dx = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2} \mathrm{d}x, \tag{A.2}$$

$$\text{step function}: \ \theta[x] = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0, \end{cases} \tag{A.3}$$

$$\text{error function}: \ \mathrm{Erf}[x] = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}t \ \mathrm{e}^{-t^2}, \tag{A.4}$$

$$\text{delta distribution}: \ \delta[x]. \tag{A.5}$$

The definition of the delta distribution will be given below.

A.2 Series and expansions

The following basic power series and expansions are being used repeatedly, both for scalar objects and for matrices

$$\mathrm{e}^x = \sum_{n\geq 0} \frac{x^n}{n!}, \tag{A.6}$$

$$(1-x)^{-1} = \sum_{n\geq 0} x^n \quad (|x| < 1), \tag{A.7}$$

$$\log(1+x) = \sum_{n\geq 1} \frac{(-1)^{n-1} x^n}{n} \quad (|x| \leq 1), \tag{A.8}$$

$$\sin(x) = x - \frac{1}{6}x^3 + \mathcal{O}(x^5) \quad (x \to 0), \tag{A.9}$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \mathcal{O}(x^4) \quad (x \to 0), \tag{A.10}$$

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\Big[1 - \frac{1}{3}x^3 + \mathcal{O}(x^5)\Big] \quad (x \to 0), \tag{A.11}$$

$$\mathrm{Erf}(x) = 1 - \frac{1}{x\sqrt{\pi}}e^{-x^2}\Big[1 + \mathcal{O}\Big(\frac{1}{x^2}\Big)\Big] \quad (|x| \to \infty). \tag{A.12}$$

A.3 The δ-Distribution

A.3.1 Definition

Here we present an intuitive definition first, and a slightly more formal one later. We first present the δ-distribution as the probability distribution $\delta(x)$ corresponding to a random variable x in the limit where the randomness in this variable vanishes. If x is 'distributed' around zero, this implies that $\int \mathrm{d}x\ f(x)\delta(x) = f(0)$ for every function f. A problem arises when we want to write down an explicit expression for $\delta(x)$. Intuitively one could think of writing

$$\delta(x) = \lim_{\Delta \to 0} G_\Delta(x), \quad G_\Delta(x) = \frac{1}{\Delta\sqrt{2\pi}}\mathrm{e}^{-(1/2)x^2/\Delta^2}. \tag{A.13}$$

However, the result is not a true function in a mathematical sense, since $\delta(x)$ is zero for $x \neq 0$ and $\delta(0) = \infty$. The way to interpret and use expressions like (A.13) is to realize that $\delta(x)$ only has a meaning when appearing inside an integration; one is to take the limit $\Delta \to 0$ only *after* performing this integration. Upon adopting this convention, we can use (A.13) to derive the following properties for sufficiently well-behaved and differentiable functions f:[73]

$$\int \mathrm{d}x\ \delta(x)f(x) = \lim_{\Delta \to 0}\int \mathrm{d}x\ G_\Delta(x)f(x)$$
$$= \lim_{\Delta \to 0}\int \frac{\mathrm{d}x}{\sqrt{2\pi}}\ \mathrm{e}^{-(1/2)x^2} f(\Delta x) = f(0), \tag{A.14}$$

$$\int \mathrm{d}x\ \delta'(x)f(x) = \lim_{\Delta \to 0}\int \mathrm{d}x \left\{\frac{\mathrm{d}}{\mathrm{d}x}[G_\Delta(x)f(x)] - G_\Delta(x)f'(x)\right\}$$
$$= \lim_{\Delta \to 0}\ [G_\Delta(x)f(x)]_{-\infty}^{\infty} - f'(0) = -f'(0). \tag{A.15}$$

[73] The precise conditions on the so-called 'test-functions' f can be properly formalized, which is the subject of distribution theory. Here we just concentrate on basic ideas and properties.

These properties can be generalized to

$$\int \mathrm{d}x\; f(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n}\delta(x) = (-1)^n \lim_{x\to 0}\frac{\mathrm{d}^n}{\mathrm{d}x^n}f(x) \quad (n = 0, 1, 2, \ldots). \tag{A.16}$$

We may now take the properties (A.16) as our *definition* of the δ-distribution.

A.3.2 Representations, relations, and generalizations

We can use the definitions of Fourier transforms $\mathcal{F}{:}f \to \hat{f}$ and inverse Fourier transforms $\mathcal{F}^{-1}{:}\hat{f} \to f$, acting in the function space $L^2(\mathbb{R})$ and defined as

$$\hat{f}(k) = \int \mathrm{d}x\; \mathrm{e}^{-2\pi ikx} f(x), \quad f(x) = \int \mathrm{d}k\; \mathrm{e}^{2\pi ikx}\hat{f}(k)$$

to obtain an integral representation of the δ-distribution. Upon writing the operator identity $\mathbb{I} = \mathcal{F}\mathcal{F}^{-1}$ in explicit form we find the general relation

$$f(x) = \int \mathrm{d}k\; \mathrm{e}^{2\pi ikx} \int \mathrm{d}y\; \mathrm{e}^{-2\pi iky} f(y).$$

Application to $f(x) = \delta(x)$ subsequently gives our desired result

$$\delta(x) = \int \mathrm{d}k\; \mathrm{e}^{2\pi ikx} = \int \frac{\mathrm{d}k}{2\pi}\, \mathrm{e}^{ikx}. \tag{A.17}$$

It is equally easy to construct a representation of the δ-distribution in the form of an infinite sum

$$\omega \in [-\pi, \pi]: \quad \delta(\omega) = \frac{1}{2\pi}\sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega\ell}. \tag{A.18}$$

To confirm the validity of (A.18), we first show that the sum in (A.18) vanishes for $\omega \neq 0$:

$$\begin{aligned}\sum_{\ell=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega\ell} &= \sum_{\ell\geq 0}\left[\mathrm{e}^{\mathrm{i}\omega\ell} + \mathrm{e}^{-\mathrm{i}\omega\ell}\right] - 1 = \frac{1}{1-\mathrm{e}^{\mathrm{i}\omega}} + \frac{1}{1-\mathrm{e}^{-\mathrm{i}\omega}} - 1\\ &= \frac{2 - \mathrm{e}^{-\mathrm{i}\omega} - \mathrm{e}^{\mathrm{i}\omega} - |1-\mathrm{e}^{\mathrm{i}\omega}|^2}{|1-\mathrm{e}^{\mathrm{i}\omega}|^2} = 0.\end{aligned}$$

Finally we have to verify that the singularity of the sum at $\omega = 0$ indeed gives us exactly the normalization required of the δ-distribution

$$\int_{-\pi}^{\pi}\mathrm{d}\omega\Big[\frac{1}{2\pi}\sum_{\ell=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega\ell}\Big] = \sum_{\ell=-\infty}^{\infty}\int_{-\pi}^{\pi}\frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}\omega\ell} = \sum_{\ell=-\infty}^{\infty}\delta_{\ell,0} = 1.$$

The following two identities relate the δ-distribution to the step function, and reveal the effect of performing continuously differentiable and invertible transformations g inside a δ-distribution, respectively:

$$\delta(x) = \frac{\mathrm{d}}{\mathrm{d}x}\theta(x), \tag{A.19}$$

$$\delta\left[g(x)-g(y)\right]=\frac{\delta(x-y)}{|g'(y)|}. \tag{A.20}$$

Both (A.19) and (A.20) are proven by showing that both sides of the equality have the same effect inside an integration, for any arbitrary test-function f:

$$\begin{aligned}\int \mathrm{d}x\Big\{\delta(x)-\frac{\mathrm{d}}{\mathrm{d}x}\theta(x)\Big\}f(x)&=f(0)-\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\mathrm{d}x\Big\{\frac{\mathrm{d}}{\mathrm{d}x}[\theta(x)f(x)]-f'(x)\theta(x)\Big\}\\&=f(0)-\lim_{\epsilon\to 0}[f(\epsilon)-0]+\lim_{\epsilon\to 0}\int_0^{\epsilon}\mathrm{d}x\, f'(x)=0.\end{aligned}$$

Similarly (noting the extra term $\mathrm{sgn}[g'(x)]$, which reflects the possible exchange of integration boundaries):

$$\begin{aligned}\int_a^b \mathrm{d}x\Big\{\delta[g(x)-g(y)]-\frac{\delta(x-y)}{|g'(y)|}\Big\}f(x)&=\int_a^b \mathrm{d}x\, g'(x)\delta[g(x)-g(y)]\frac{f(x)}{g'(x)}-\frac{f(y)}{|g'(y)|}\\&=\int_{g(a)}^{g(b)}\mathrm{d}u\,\delta[u-g(y)]\frac{f(g^{-1}(u))}{g'(g^{-1}(u))}-\frac{f(y)}{|g'(y)|}\\&=\frac{f(y)}{|g'(y)|}-\frac{f(y)}{|g'(y)|}=0.\end{aligned}$$

Finally, the generalization to $\mathbf{x}=(x_1,\ldots,x_N)\in\mathbb{R}^N$ is straightforward:

$$\delta(\mathbf{x})=\prod_{i=1}^{N}\delta(x_i). \tag{A.21}$$

A.4 Steepest descent integration

The objective of steepest descent (or ‘saddle-point’) integration in its simplest form is to deal with integrals of the following type, with $\mathbf{x}\in\mathbb{R}^p$, with continuous functions $f(\mathbf{x})$ and $g(\mathbf{x})$, of which f is bounded from below, and with $N\in\mathbb{R}$ positive and large

$$I_N[f,g]=\int_{\mathbb{R}^p}\mathrm{d}\mathbf{x}\, g(\mathbf{x})\mathrm{e}^{-Nf(\mathbf{x})}. \tag{A.22}$$

We first take $f(\mathbf{x})$ to be real-valued; this is the simplest case, for which finding the asymptotic behaviour of (A.22) as $N\to\infty$ goes back to Laplace. We assume that

$f(\mathbf{x})$ can be expanded in a Taylor series around its minimum $f(\mathbf{x}^\star)$, which we assume to be unique, i.e.

$$f(\mathbf{x}) = f(\mathbf{x}^\star) + \frac{1}{2}\sum_{ij=1}^{p} A_{ij}(x_i - x_i^\star)(x_j - x_j^\star) + \mathcal{O}(|\mathbf{x}-\mathbf{x}^\star|^3), \quad A_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}|_{\mathbf{x}^\star}. \tag{A.23}$$

Provided the integral (A.22) exists, insertion of (A.23) into (A.22) followed by transformation $\mathbf{x} = \mathbf{x}^\star + \mathbf{y}/\sqrt{N}$ gives

$$\begin{aligned} I_N[f,g] &= \mathrm{e}^{-Nf(\mathbf{x}^\star)} \int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\, g(\mathbf{x}) \mathrm{e}^{-(1/2)N\sum_{ij}(x_i - x_i^\star)A_{ij}(x_j - x_j^\star) + \mathcal{O}(N|\mathbf{x}-\mathbf{x}^\star|^3)} \\ &= N^{-(p/2)}\mathrm{e}^{-Nf(\mathbf{x}^\star)} \int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\, g\left(\mathbf{x}^\star + \frac{\mathbf{y}}{\sqrt{N}}\right)\, \mathrm{e}^{-(1/2)\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(N^{-(1/2)}|\mathbf{y}|^3)}. \end{aligned} \tag{A.24}$$

From this latter expansion, and given the assumptions made, we can obtain two important identities, both of which are being used regularly in the present book

$$\begin{aligned} -\lim_{N\to\infty} \frac{1}{N} \log \int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\, \mathrm{e}^{-Nf(\mathbf{x})} &= -\lim_{N\to\infty} \frac{1}{N} \log I_N[f,1] \\ = f(\mathbf{x}^\star) + \lim_{N\to\infty} \left\{ \frac{p\log N}{2N} - \frac{1}{N}\log \int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\, \mathrm{e}^{-(1/2)\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(N^{-(1/2)}|\mathbf{y}|^3)} \right\} \\ = f(\mathbf{x}^\star) = \min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}) \end{aligned} \tag{A.25}$$

and

$$\begin{aligned} \lim_{N\to\infty} \frac{\int \mathrm{d}\mathbf{x}\, g(\mathbf{x})\mathrm{e}^{-Nf(\mathbf{x})}}{\int \mathrm{d}\mathbf{x}\, \mathrm{e}^{-Nf(\mathbf{x})}} &= \lim_{N\to\infty} \frac{I_N[f,g]}{I_N[f,1]} \\ &= \lim_{N\to\infty} \left\{ \frac{\int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\, g(\mathbf{x}^\star + \frac{\mathbf{y}}{\sqrt{N}})\, \mathrm{e}^{-(1/2)\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(N^{-(1/2)})|\mathbf{y}|^3)}}{\int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\, \mathrm{e}^{-(1/2)\sum_{ij} y_i A_{ij} y_j + \mathcal{O}(N^{-(1/2)}|\mathbf{y}|^3)}} \right\} \\ &= \frac{g(\mathbf{x}^\star)(2\pi)^{p/2}/\sqrt{\mathrm{Det}A}}{(2\pi)^{p/2}/\sqrt{\mathrm{Det}A}} = g(\mathbf{x}^\star) \\ &= g(\arg \min_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x})). \end{aligned} \tag{A.26}$$

The situation becomes more complicated when we allow the dimension p of our integrations to depend on N. Provided the ratio p/N goes to zero sufficiently fast as $N \to \infty$, one can still prove the above identities, but much more care will be needed in dealing with correction terms.

In those cases where the function $f(\mathbf{x})$ is complex, it is much less clear how to calculate the asymptotic form of (A.22). The correct procedure to be followed here is to deform the integration paths in the complex plane (using Cauchy's theorem) such that along the deformed path the imaginary part of the function $f(\mathbf{x})$ is constant, and preferably (if possible) zero. In the models analyzed in this book we can be sure on physical grounds that a path where $f(\mathbf{x}) \in \mathbb{R}$ can always be found. Having succeeded, one can then proceed using Laplace's argument and find the leading order in N of our integral in the usual manner by extremization of the real part of $f(\mathbf{x})$. In combination one finds that our integrals will thus again be dominated by an extremum of the (complex) function $f(\mathbf{x})$, but since f is generally complex we can no longer interpret this extremum as a minimum

$$-\lim_{N\to\infty} \frac{1}{N} \log \int_{\mathbb{R}^p} \mathrm{d}\mathbf{x}\ \mathrm{e}^{-Nf(\mathbf{x})} = \mathrm{extr}_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x}), \tag{A.27}$$

$$\lim_{N\to\infty} \frac{\int \mathrm{d}\mathbf{x}\ g(\mathbf{x})\mathrm{e}^{-Nf(\mathbf{x})}}{\int \mathrm{d}\mathbf{x}\ \mathrm{e}^{-Nf(\mathbf{x})}} = g(\arg\ \mathrm{extr}_{\mathbf{x}\in\mathbb{R}^p} f(\mathbf{x})). \tag{A.28}$$

Whereas the Laplace method for real functions $f(\mathbf{x})$ is relatively straightforward, the precise mathematical details of the full saddle-point method (including the subtleties in the correct deformation of integration paths, the question of under which conditions a path with $f(\mathbf{x}) \in \mathbb{R}$ exists, etc.) are more involved. For these the reader will therefore have to be referred to textbooks on methods in mathematical physics.

APPENDIX B: Integrals

B.1 Gaussian integrals

B.1.1 Properties of real and positive definite matrices

Let us consider symmetric, real-valued and positive definite $N \times N$ matrices $\mathbf{A}$, i.e. $\mathbf{x} \cdot \mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^N$ with $|\mathbf{x}| > 0$. The eigenvalue polynomial $\det[\mathbf{A} - \lambda \mathbb{1}] = 0$ is of order N, so the eigenvalue problem $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$ will always have N (possibly complex) solutions λ (of which some may coincide). We denote the complex conjugate of a complex number $z = a + \mathrm{i}b$ as $z^* = a - \mathrm{i}b$, and define the unit matrix $\mathbb{1}$ with entries $\mathbb{1}_{ij} = \delta_{ij}$.

We will use the following properties, proofs of which are found in every linear algebra textbook: (i) all eigenvalues λ_i of the matrix $\mathbf{A}$ are real and positive, (ii) all eigenvectors $\mathbf{x}$ can be chosen real-valued, and (iii) we can construct a complete orthogonal basis in $\mathbb{R}^N$ out of eigenvectors of $\mathbf{A}$. Thus there exists a set of N vectors $\hat{\mathbf{e}}^i = (e_1^i, \ldots, e_N^i) \in \mathbb{R}^N$, $i = 1, \ldots, N$, with the properties

$$\mathbf{A}\hat{\mathbf{e}}^i = \lambda_i \hat{\mathbf{e}}^i, \quad \lambda_i \in \mathbb{R},\ \lambda_i > 0, \quad \hat{\mathbf{e}}^i \cdot \hat{\mathbf{e}}^j = \delta_{ij}.$$

We construct a new matrix $\mathbf{U}$ from the components of the normalized eigenvectors $\{\hat{\mathbf{e}}^i\}$, according to $U_{ij} = \hat{e}_i^j$. We denote its transpose by $\mathbf{U}^\dagger$, i.e. $U_{ij}^\dagger = U_{ji}$, and show that $\mathbf{U}$ is unitary, i.e. that $\mathbf{U}^\dagger \mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbb{1}$:

$$\sum_j (\mathbf{U}^\dagger \mathbf{U})_{ij} x_j = \sum_{jk} U^{ki} U_{kj} x_j = \sum_{jk} \hat{e}_k^i \hat{e}_k^j x_j = \sum_j \delta_{ij} x_j = x_i$$
$$\sum_j (\mathbf{U}\mathbf{U}^\dagger)_{ij} x_j = \sum_{jk} U^{ik} U_{jk} x_j = \sum_{jk} \hat{e}_i^k \hat{e}_j^k x_j = \sum_k \hat{e}_i^k (\hat{\mathbf{e}} \cdot \mathbf{x}) = x_i$$

(since $\{\hat{\mathbf{e}}^\ell\}$ forms a complete orthogonal basis). The unitary matrix $\mathbf{U}$ is seen to bring $\mathbf{A}$ onto diagonal form:

$$(\mathbf{U}^\dagger \mathbf{A} \mathbf{U})_{ij} = \sum_{kl=1}^N U_{ik}^\dagger A_{kl} U_{lj} = \sum_{kl=1}^N \hat{e}_k^i A_{kl} \hat{e}_l^j = \lambda_j \sum_{k=1}^N \hat{e}_k^i \hat{e}_k^j = \lambda_j \delta_{ij}. \qquad \text{(B.1)}$$

Finally, the inverse $\mathbf{A}^{-1}$ of the matrix $\mathbf{A}$ always exists, and its entries are

$$(\mathbf{A}^{-1})_{ij} = \sum_{k=1}^{N} \lambda_k^{-1} \hat{e}_i^k \hat{e}_j^k, \tag{B.2}$$

(which is easily confirmed by substitution into the products $\mathbf{A}\mathbf{A}^{-1}$ and $\mathbf{A}^{-1}\mathbf{A}$).

B.1.2 Multi-variate Gaussian integrals

We now use the above properties to calculate those associated Gaussian integrals which play a role in this book, all of the general form $I = \int \mathrm{d}\mathbf{x}\ f(\mathbf{x})\mathrm{e}^{-(1/2)\mathbf{x}\cdot\mathbf{A}\mathbf{x}}$, with $\mathbf{x} \in \mathbb{R}^N$, with $\mathbf{A}$ denoting a positive definite real-valued symmetric matrix (with $\mathbf{A}\hat{\mathbf{e}}^\ell = \lambda_\ell \hat{\mathbf{e}}^\ell$ and $\hat{\mathbf{e}}^i \cdot \hat{\mathbf{e}}^j = \delta_{ij}$), and with $\mathbf{U}$ the associated unitary diagonalization matrix in the sense of the previous sub-section

$$\begin{aligned}\int \mathrm{d}x\ \mathrm{e}^{-(1/2)x^2} &= \left\{\int \mathrm{d}x_1 \mathrm{d}x_2\ \mathrm{e}^{-(1/2)(x_1^2+x_2^2)}\right\}^{1/2} = \left\{\int_0^\infty \mathrm{d}r\ r \int_0^{2\pi} \mathrm{d}\phi\ \mathrm{e}^{-(1/2)r^2}\right\}^{\frac{1}{2}} \\ &= \left\{2\pi\left[-\mathrm{e}^{-(1/2)r^2}\right]_0^\infty\right\}^{1/2} = \sqrt{2\pi},\end{aligned} \tag{B.3}$$

$$\begin{aligned}\int \mathrm{d}\mathbf{x}\ \mathrm{e}^{-(1/2)\mathbf{x}\cdot\mathbf{A}\mathbf{x}+\mathbf{b}\cdot\mathbf{x}} &= \int \mathrm{d}\mathbf{U}\mathbf{z}\ \mathrm{e}^{-(1/2)\mathbf{z}\cdot\mathbf{U}^\dagger\mathbf{A}\mathbf{U}\mathbf{z}+\mathbf{z}\cdot\mathbf{U}^\dagger\mathbf{b}} \\ &= \prod_{\ell=1}^{N}\left[\int \mathrm{d}z\ \mathrm{e}^{-(1/2)\lambda_\ell z^2+z(\mathbf{U}^\dagger\mathbf{b})_\ell}\right] \\ &= \prod_{\ell=1}^{N}\left[\int \mathrm{d}z\ \mathrm{e}^{-(1/2)\lambda_\ell[z-(\mathbf{U}^\dagger\mathbf{b})_\ell\lambda_\ell^{-1}]^2+(1/2)\lambda_\ell^{-1}(\mathbf{U}^\dagger\mathbf{b})_\ell^2}\right] \\ &= \mathrm{e}^{(1/2)\sum_{ij\ell=1}^{N}\lambda_\ell^{-1}U_{i\ell}b_iU_{j\ell}b_j}\prod_{\ell=1}^{N}\left[\int \mathrm{d}z\ \mathrm{e}^{-(1/2)\lambda_\ell z^2}\right] \\ &= \frac{(2\pi)^{N/2}}{\sqrt{\det\mathbf{A}}}\mathrm{e}^{(1/2)\sum_{ij=1}^{N}b_ib_j\sum_{\ell=1}^{N}\lambda_\ell^{-1}\hat{e}_i^\ell\hat{e}_j^\ell} \\ &= \frac{(2\pi)^{N/2}}{\sqrt{\det\mathbf{A}}}\mathrm{e}^{(1/2)\mathbf{b}\cdot\mathbf{A}^{-1}\mathbf{b}},\end{aligned} \tag{B.4}$$

$$\begin{aligned}&\int \mathrm{d}\mathbf{x}\ x_i \mathrm{e}^{-(1/2)\mathbf{x}\cdot\mathbf{A}\mathbf{x}+\mathbf{b}\cdot\mathbf{x}} \\ &\quad = \sum_j U_{ij}\int \mathrm{d}\mathbf{U}\mathbf{z}\ z_j \mathrm{e}^{-(1/2)\mathbf{z}\cdot\mathbf{U}^\dagger\mathbf{A}\mathbf{U}\mathbf{z}+\mathbf{z}\cdot\mathbf{U}^\dagger\mathbf{b}}\end{aligned}$$

$$
\begin{aligned}
&= \sum_j U_{ij} \left\{ \int \mathrm{d}z\ z\ \mathrm{e}^{-(1/2)\lambda_j z^2 + z(\mathbf{U}^\dagger \mathbf{b})_j} \prod_{\ell \neq j} \left[\int \mathrm{d}z\ \mathrm{e}^{-(1/2)\lambda_\ell z^2 + z(\mathbf{U}^\dagger \mathbf{b})_\ell} \right] \right\} \\
&= \sum_j U_{ij} (\mathbf{U}^\dagger \mathbf{b})_j \lambda_\ell^{-1} \mathrm{e}^{(1/2)\sum_{pk\ell=1}^N \lambda_\ell^{-1} U_{p\ell} b_p U_{k\ell} b_k} \prod_{\ell=1}^N \left[\int \mathrm{d}z\ \mathrm{e}^{-(1/2)\lambda_\ell z^2} \right] \\
&= \sum_{jk} U_{ij} U_{jk}^\dagger b_k \lambda_j^{-1} \frac{(2\pi)^{N/2}}{\sqrt{\det \mathbf{A}}} \mathrm{e}^{(1/2)\sum_{pk=1}^N b_p b_k \sum_{\ell=1}^N \lambda_\ell^{-1} \hat{e}_p^\ell \hat{e}_k^\ell} \\
&= (\mathbf{A}^{-1}\mathbf{b})_i \frac{(2\pi)^{N/2}}{\sqrt{\det \mathbf{A}}} \mathrm{e}^{1/2 \mathbf{b}\cdot\mathbf{A}^{-1}\mathbf{b}}. \qquad \text{(B.5)}
\end{aligned}
$$

In deriving (B.4) and (B.5) we used (B.3), as well as the identity $\prod_{\ell=1}^N \lambda_\ell = \det \mathbf{A}$. From (B.4), in turn, follow for instance

$$
\int \mathrm{d}\mathbf{x}\ \mathrm{e}^{-(1/2)\mathbf{x}\cdot\mathbf{A}\mathbf{x}} = \frac{(2\pi)^{N/2}}{\sqrt{\det \mathbf{A}}}, \qquad \text{(B.6)}
$$

$$
\begin{aligned}
\int \mathrm{d}\mathbf{x}\ x_i x_j \mathrm{e}^{-(1/2)\mathbf{x}\cdot\mathbf{A}\mathbf{x}} &= \lim_{\mathbf{b}\to\mathbf{0}} \frac{\partial^2}{\partial b_i \partial b_j} \int \mathrm{d}\mathbf{x}\ \mathrm{e}^{-(1/2)\mathbf{x}\cdot\mathbf{A}\mathbf{x} + \mathbf{b}\cdot\mathbf{x}} \\
&= \lim_{\mathbf{b}\to\mathbf{0}} \frac{\partial^2}{\partial b_i \partial b_j} \frac{(2\pi)^{N/2}}{\sqrt{\det \mathbf{A}}} \mathrm{e}^{(1/2)\mathbf{b}\cdot\mathbf{A}^{-1}\mathbf{b}} = \frac{(\mathbf{A}^{-1})_{ij} (2\pi)^{N/2}}{\sqrt{\det \mathbf{A}}}. \qquad \text{(B.7)}
\end{aligned}
$$

Combination of (B.6) and (B.7) shows that

$$
\frac{\int \mathrm{d}\mathbf{x}\ x_i x_j \mathrm{e}^{-(1/2)\mathbf{x}\cdot\mathbf{A}\mathbf{x}}}{\int \mathrm{d}\mathbf{x}\ \mathrm{e}^{-(1/2)\mathbf{x}\cdot\mathbf{A}\mathbf{x}}} = (\mathbf{A}^{-1})_{ij}. \qquad \text{(B.8)}
$$

B.2 Integrals in the replica analysis

In Chapter 3 and most of the subsequent chapters we encounter a number of specific integrals. The first three are

$$
\int_0^{A\sqrt{2}} Dx = \frac{1}{\sqrt{\pi}} \int_0^A \mathrm{d}t\ \mathrm{e}^{-t^2} = \frac{1}{2} \mathrm{Erf}[A], \qquad \text{(B.9)}
$$

$$
\int_0^{A\sqrt{2}} Dx\ x = \frac{1}{\sqrt{2\pi}} \Big[-\mathrm{e}^{-(1/2)x^2} \Big]_0^{A\sqrt{2}} = \frac{1}{\sqrt{2\pi}} \Big[1 - \mathrm{e}^{-(1/2)A^2} \Big] \qquad \text{(B.10)}
$$

$$
\int_0^{A\sqrt{2}} Dx\ x^2 = \frac{-1}{\sqrt{2\pi}} \lim_{u\to 1/2} \frac{\mathrm{d}}{\mathrm{d}u} \int_0^{A\sqrt{2}} \mathrm{d}x\ \mathrm{e}^{-ux^2}
$$

$$= \frac{-1}{\sqrt{2\pi}} \lim_{u\to 1/2} \frac{\mathrm{d}}{\mathrm{d}u} \left\{ u^{-1/2} \int_0^{A\sqrt{2u}} \mathrm{d}x\ \mathrm{e}^{-x^2} \right\}$$
$$= \frac{1}{2}\mathrm{Erf}[A] - \frac{A}{\sqrt{\pi}} \mathrm{e}^{-A^2}. \tag{B.11}$$

We also regularly need integrals involving the following function $s(a)$ and its derivative, with $\lambda \geq 0$:

$$s(a) = \begin{cases} a & \text{for } |a| \leq \lambda \\ \lambda\ \mathrm{sgn}[a] & \text{for } |a| > \lambda, \end{cases} \qquad s'(a) = \begin{cases} 1 & \text{for } |a| \leq \lambda \\ 0 & \text{for } |a| > \lambda. \end{cases} \tag{B.12}$$

The three relevant integrals involving this function $s(a)$ are the following (with $u > 0$ and $0 < \epsilon < 1$):

$$\int Dx\ s^2\left(\frac{x}{u\sqrt{2}}\right) = \lambda^2 + \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{u\sqrt{2}} \mathrm{d}x\ \mathrm{e}^{-(1/2)x^2} \left\{ \frac{x^2}{2u^2} - \lambda^2 \right\}$$
$$= \lambda^2 \left\{ 1 - \mathrm{Erf}[\lambda u] - \frac{1}{\lambda^2\sqrt{\pi}} \frac{\mathrm{d}}{\mathrm{d}u} \int_0^{\lambda} \mathrm{d}x\ \mathrm{e}^{-u^2x^2} \right\}$$
$$= \lambda^2 \left\{ 1 + \frac{1 - 2\lambda^2u^2}{2\lambda^2u^2} \mathrm{Erf}[\lambda u] - \frac{1}{\lambda u\sqrt{\pi}} \mathrm{e}^{-\lambda^2u^2} \right\}, \tag{B.13}$$

$$\int Dx\ s'(\frac{x}{u\sqrt{2}}) = \int_{-\lambda u\sqrt{2}}^{\lambda u\sqrt{2}} Dx\ 1 = \mathrm{Erf}[\lambda u], \tag{B.14}$$

$$\int Dx\ \theta\Big[s^2(\frac{x}{u\sqrt{2}}) - \epsilon^2\lambda^2\Big] = 2\int_{\epsilon\lambda u\sqrt{2}}^{\infty} Dx\ 1 = 1 - \mathrm{Erf}[\epsilon\lambda u]. \tag{B.15}$$

B.3 Integrals in the generating functional analysis

In the chapter on the batch MG with fake market information we first encounter the following integral:

$$\int Dx\ \mathrm{Erf}(A + Bx) = \int_0^A \mathrm{d}y \int Dx\ \frac{2}{\sqrt{\pi}} \mathrm{e}^{-(y+Bx)^2}$$
$$= \frac{2}{\sqrt{\pi}} \int_0^A \mathrm{d}y\ \mathrm{e}^{-y^2} \int \frac{\mathrm{d}x}{\sqrt{2\pi}} \mathrm{e}^{-(1/2)x^2(1+2B^2)-2Byx}$$

$$= \frac{\sqrt{2}}{\pi\sqrt{1+2B^2}} \int_0^A \mathrm{d}y\ \mathrm{e}^{-y^2}$$
$$\times \int \mathrm{d}x\ \mathrm{e}^{-(1/2)(x+(2By/\sqrt{1+2B^2}))^2+2B^2y^2/(1+2B^2)}$$
$$= \frac{2}{\sqrt{\pi}\sqrt{1+2B^2}} \int_0^A \mathrm{d}y\ \mathrm{e}^{-(y^2/1+2B^2)} = \mathrm{Erf}\left[\frac{A}{\sqrt{1+2B^2}}\right]. \quad \text{(B.16)}$$

The following integral is needed in the evaluation of the relative history frequency $\varrho(f)$ in Chapter 8:

$$\int Dx\ \mathrm{Erf}^2(Ax) = \int_0^A \mathrm{d}y\ \frac{\partial}{\partial y} \int Dx\ \mathrm{Erf}^2(yx)$$
$$= \int_0^A \mathrm{d}y \int Dx\ x\ \mathrm{Erf}(yx) \frac{4}{\sqrt{\pi}} \mathrm{e}^{-y^2x^2}$$
$$= \frac{4}{\sqrt{\pi}} \int_0^A \mathrm{d}y \int \frac{\mathrm{d}x}{\sqrt{2\pi}} \mathrm{e}^{-(1/2)x^2(1+2y^2)}\ x\ \mathrm{Erf}(yx)$$
$$= \frac{4}{\sqrt{\pi}} \int_0^A \frac{\mathrm{d}y}{1+2y^2} \int \frac{\mathrm{d}x}{\sqrt{2\pi}} \mathrm{e}^{-(1/2)x^2}\ \frac{\partial}{\partial x}\ \mathrm{Erf}\left(\frac{xy}{\sqrt{1+2y^2}}\right)$$
$$= \frac{8}{\pi} \int_0^A \frac{\mathrm{d}y\ y}{(1+2y^2)^{3/2}} \int \frac{\mathrm{d}x}{\sqrt{2\pi}} \mathrm{e}^{-(1/2)x^2-x^2y^2/(1+2y^2)}$$
$$= \frac{8}{\pi} \int_0^A \frac{\mathrm{d}y\ y}{(1+2y^2)\sqrt{1+4y^2}}$$
$$= \frac{2}{\pi} \int_1^{1+4A^2} \frac{\mathrm{d}t}{(1+t)\sqrt{t}} = \frac{4}{\pi}\Big[\arctan(\sqrt{t})\Big]_1^{1+4A^2}$$
$$= \frac{4}{\pi} \arctan(\sqrt{1+4A^2}) - 1. \quad \text{(B.17)}$$

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APPENDIX C: Moments of random matrices

In this appendix we derive the expressions used in Chapter 8 for some of the moments of the random matrices defined by (8.54). We abbreviate $\overline{\delta}_{kl} = 1 - \delta_{kl}$ and $\langle Q[\mathcal{B}] \rangle_{\mathcal{B}} = \int\!\mathrm{d}\mathcal{B}\, \mathcal{P}[\mathcal{B}] Q[\mathcal{B}]$. The random matrix ensemble is defined as

$$\mathcal{P}[\mathcal{B}] = \left\langle \prod_{\ell,\ell'} \delta\Big[\mathcal{B}_{\ell\ell'} - \prod_{\lambda=1}^{\log_2(p)} \theta\left[Z(\ell,\lambda) Z(\ell',\lambda)\right]\Big] \right\rangle_{\{Z\}}. \tag{C.1}$$

The $\{Z(\ell,\lambda)\}$ are zero-average Gaussian random variables, representing inconsistent 'fake memory', characterized by

$$\langle Z(\ell,\lambda) Z(\ell',\lambda') \rangle_{\{Z\}} = \kappa^2 \delta_{\ell\ell'} \delta_{\lambda\lambda'}. \tag{C.2}$$

Thus $\mathcal{B}_{\ell\ell'} \in \{0,1\}$, and $\langle \mathcal{B}_{kl}^2 \rangle = \langle \mathcal{B}_{kl} \rangle$. Averages come out as

$$\begin{aligned} \langle \mathcal{B}_{kl} \rangle &= \prod_{\lambda=1}^{\log_2(p)} \langle \theta\left[Z(k,\lambda) Z(\ell,\lambda)\right] \rangle_{\{Z\}} = \delta_{k\ell} + \overline{\delta}_{k\ell} \left(\frac{1}{2}\right)^{\log_2(p)} \\ &= \delta_{k\ell} + \frac{1}{p} \overline{\delta}_{k\ell}. \end{aligned} \tag{C.3}$$

Our analysis in Chapter 8 involves two further types of averages. The first is a cyclic combination of rth moments, where $s_1 > s_2 > \cdots > s_r$ and $r > 1$ (no summations):

$$\begin{aligned} &\langle \mathcal{B}_{s_1 s_2} \mathcal{B}_{s_2,s_3} \cdots \mathcal{B}_{s_r,s_1} \rangle_{\mathcal{B}} \\ &\quad = \prod_{\lambda=1}^{\log_2(p)} \langle \theta[Z(s_1,\lambda) Z(s_2,\lambda)] \theta[Z(s_2,\lambda) Z(s_3,\lambda)] \cdots \theta[Z(s_r,\lambda) Z(s_1,\lambda)] \rangle_{\{Z\}} \\ &\quad = \left\{ \langle \theta[Z(s_1) Z(s_2)] \theta[Z(s_1) Z(s_3)] \cdots \theta[Z(s_1) Z(s_r)] \rangle_{\{Z\}} \right\}^{\log_2(p)} \\ &\quad = 2^{(1-r)\log_2(p)} = p^{1-r}. \end{aligned} \tag{C.4}$$

We note that (C.4) also holds trivially when $r = 1$. The second non-trivial average is formed of two time-ordered strings of matrix elements (of lengths r and r', respectively) connected by two further matrix elements, where $s_0 > s_1 > \cdots > s_r$ and

$s'_0 > s'_1 > \cdots > s'_{r'}$. Upon dropping the index λ as soon as it has become irrelevant (to reduce the size of our equations) and upon renaming $s_{r+1} = k$ and $s_{r'+1} = k'$ we get

$$\begin{aligned}
&\langle [\mathcal{B}_{s_0 s_1}\mathcal{B}_{s_1 s_2}\cdots\mathcal{B}_{s_{r-1}s_r}]\mathcal{B}_{s_r s'_{r'}}[\mathcal{B}_{s'_0 s'_1}\mathcal{B}_{s'_1 s'_2}\cdots\mathcal{B}_{s'_{r'-1}s'_{r'}}]\mathcal{B}_{s_0 s'_0}\rangle \\
&\quad = \Big\langle \theta[Z(s_0)Z(s_1)]\theta[Z(s_1)Z(s_2)]\cdots\theta[Z(s_{r-1})Z(s_r)]\;\theta[Z(s_r)Z(s'_{r'})] \\
&\qquad \times \theta[Z(s'_0)Z(s'_1)]\theta[Z(s'_1)Z(s'_2)]\ldots\theta[Z(s'_{r'-1})Z(s'_{r'})]\;\theta[Z(s_0)Z(s'_0)]\Big\rangle_{\{Z\}}^{\log_2(p)}.
\end{aligned} \tag{C.5}$$

It is clear that the time indices in (C.5) can be at most pairwise identical, with indices of the string $\{s_0, \cdots, s_r\}$ coinciding with indices of the string $\{s'_0, \cdots, s'_{r'}\}$. The number of such pairings is given by $\Gamma = \sum_{i=0}^{r}\sum_{j=0}^{r'}\delta_{s_i s'_j}$, leaving in (C.5) an average involving only $r + r' + 2 - \Gamma$ non-identical Gaussian variables, on which the remaining condition is that that must all be of the same sign. Hence we obtain

$$\begin{aligned}
&\langle [\mathcal{B}_{s_0 s_1}\mathcal{B}_{s_1 s_2}\cdots\mathcal{B}_{s_{r-1}s_r}]\mathcal{B}_{s_r s'_{r'}}[\mathcal{B}_{s'_0 s'_1}\mathcal{B}_{s'_1 s'_2}\cdots\mathcal{B}_{s'_{r'-1}s'_{r'}}]\mathcal{B}_{s_0 s'_0}\rangle \\
&\quad = \left\{(\frac{1}{2})^{r+r'+1-\Gamma}\right\}^{\log_2(p)} = p^{\sum_{i=0}^{r}\sum_{j=0}^{r'}\delta_{s_i s'_j} - r - r' - 1}.
\end{aligned} \tag{C.6}$$

APPENDIX D: Expansion of bid sign recurrence probabilities

In this appendix we derive the expansion (8.109) of the function $\Phi(g_1, g_2, \ldots)$ as defined in (8.102). We will abbreviate

$$E_{\boldsymbol{\lambda}} = \text{Erf}\left[\frac{(1-\zeta)\overline{A}_{\boldsymbol{\lambda}}}{\sqrt{2}\sqrt{\zeta^2\kappa^2 + (1-\zeta)^2\sigma_{\boldsymbol{\lambda}}^2}}\right] \tag{D.1}$$

with $\sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}} E_{\boldsymbol{\lambda}}^r = \langle E^r \rangle$. These shorthands allow us to compactify (8.102) to

$$\begin{aligned}
\Phi(g_1, g_2, \ldots) &= \frac{1}{2}\prod_{j\geq 1}\left[\sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}}(1+E_{\boldsymbol{\lambda}})^{g_j}\right] + \frac{1}{2}\prod_{j\geq 1}\left[\sum_{\boldsymbol{\lambda}} \pi_{\boldsymbol{\lambda}}(1-E_{\boldsymbol{\lambda}})^{g_j}\right] \\
&= \frac{1}{2}\prod_{j\geq 1}\left[\sum_{n=0}^{g_j}\binom{g_j}{n}\langle E^n\rangle\right] + \frac{1}{2}\prod_{j\geq 1}\left[\sum_{n=0}^{g_j}\binom{g_j}{n}(-1)^n\langle E^n\rangle\right] \\
&= \sum_{n_1=0}^{g_1}\sum_{n_2=0}^{g_2}\cdots\binom{g_1}{n_1}\binom{g_2}{n_2}\cdots\frac{1}{2}[1+(-1)^{n_1+n_2+\cdots}]\langle E^{n_1}\rangle\langle E^{n_2}\rangle\cdots
\end{aligned} \tag{D.2}$$

Since the overall average bid in the MG is equally likely to be positive than negative, and since (D.1) tells us that $\text{sgn}[E_{\boldsymbol{\lambda}}] = \text{sgn}[\overline{A}_{\boldsymbol{\lambda}}]$, the moments $\langle E^r \rangle$ for even values of r will have to be zero. From this it follows that

$$\begin{aligned}
\Phi(g_1, g_2, \ldots) &= \sum_{0\leq n_1\leq g_1/2}\sum_{0\leq n_2\leq g_2/2}\cdots\binom{g_1}{2n_1}\binom{g_2}{2n_2}\cdots\langle E^{2n_1}\rangle\langle E^{2n_2}\rangle\cdots \\
&= \prod_{j\geq 1}\left[1+\sum_{1\leq n\leq g_j/2}\binom{g_j}{2n}\langle E^{2n}\rangle\right],
\end{aligned}$$

so

$$\log\Phi(g_1,g_2,\ldots)=\sum_{j\geq 1}\log\Big[1+\sum_{1\leq n\leq g_j/2}\binom{g_j}{2n}\langle E^{2n}\rangle\Big]. \tag{D.3}$$

Equation (D.3) tells us, firstly, that

$$\Phi(1,1,1,\ldots)=1. \tag{D.4}$$

For arbitrary history coincidence numbers $(g_1,g_2,\ldots)$, not necessarily all equal to one, we may expand (D.3) in the moments $\langle E^r\rangle$:

$$\begin{aligned}\log\Phi(g_1,g_2,\ldots)&=\sum_{j\geq 1}\log\Big[1+\frac{1}{2}g_j(g_j-1)\langle E^2\rangle+\frac{1}{24}g_j(g_j-1)(g_j-2)(g_j-3)\langle E^4\rangle\\&\quad+\mathcal{O}(\langle E^6\rangle)\Big]\\&=\frac{1}{2}\sum_{j\geq 1}g_j(g_j-1)\Big\{\langle E^2\rangle\\&\quad+\frac{1}{12}\Big[(g_j-2)(g_j-3)\langle E^4\rangle-3g_j(g_j-1)\langle E^2\rangle^2\Big]\Big\}+\mathcal{O}(\langle E^6\rangle).\end{aligned}$$

Finally, in leading order in E we may regard the variables $E_{\boldsymbol{\lambda}}$ as proportional to $\overline{A}_{\boldsymbol{\lambda}}$, and therefore in leading order distributed in a Gaussian manner. This implies (since $\langle E\rangle=0$) that in leading order we have $\langle E^4\rangle=3\langle E^2\rangle$. Hence

$$\begin{aligned}\log\Phi(g_1,g_2,\ldots)&=\frac{1}{2}\langle E^2\rangle\sum_{j\geq 1}g_j(g_j-1)-\frac{1}{4}\langle E^2\rangle^2\sum_{j\geq 1}g_j(g_j-1)(2g_j-3)\\&\quad+\mathcal{O}(\langle E^6\rangle).\end{aligned} \tag{D.5}$$

In principle one can take this expansion to higher orders, but this will inevitably lead to expressions that can no longer be written in terms of $\langle E^2\rangle$ only.

APPENDIX E: Combinatorics in history frequency moments

In this appendix we calculate the combinatorial factors $G_{a,b}^{k,R}$ as defined in (8.114). They can be obtained by differentiation of a simple generating function

$$\begin{aligned}
G_{a,b}^{k,R} &= R^{-k} \sum_{g_1=0}^{k} \sum_{g_2=0}^{k-g_1} \binom{k}{g_1} \binom{k-g_1}{g_2} (R-2)^{k-g_1-g_2} g_1^a g_2^b \\
&= R^{-k} \lim_{x,y\to 1} \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^a \left(y\frac{\mathrm{d}}{\mathrm{d}y}\right)^b \sum_{g_1=0}^{k} \sum_{g_2=0}^{k-g_1} \binom{k}{g_1} \binom{k-g_1}{g_2} (R-2)^{k-g_1-g_2} x^{g_1} y^{g_2} \\
&= R^{-k} \lim_{x,y\to 1} \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^a \left(y\frac{\mathrm{d}}{\mathrm{d}y}\right)^b \sum_{g_1=0}^{k} \sum_{g_2=0}^{k-g_1} \binom{k}{g_1} (R-2+y)^{k-g_1} x^{g_1} \\
&= R^{-k} \lim_{x,y\to 1} \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^a \left(y\frac{\mathrm{d}}{\mathrm{d}y}\right)^b (R-2+x+y)^k. \qquad \text{(E.1)}
\end{aligned}$$

In particular we find

$$\begin{aligned}
G_{2,0}^{k,R} &= R^{-k} \lim_{x\to 1} \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 (R-1+x)^k \\
&= R^{-k} \lim_{x\to 1} \left[kx(R-1+x)^{k-1} + k(k-1)x^2(R-1+x)^{k-2}\right] \\
&= kR^{-1} + k(k-1)R^{-2} \qquad \text{(E.2)}
\end{aligned}$$

$$\begin{aligned}
G_{3,0}^{k,R} &= R^{-k} \lim_{x\to 1} \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 (R-1+x)^k \\
&= R^{-k} \lim_{x\to 1} \Big[kx(R-1+x)^{k-1} + 3k(k-1)x^2(R-1+x)^{k-2} \\
&\quad + k(k-1)(k-2)x^3(R-1+x)^{k-3}\Big] \\
&= kR^{-1} + 3k(k-1)R^{-2} + k(k-1)(k-2)R^{-3}. \qquad \text{(E.3)}
\end{aligned}$$

$$\begin{aligned}
G_{4,0}^{k,R} &= R^{-k} \lim_{x\to 1} \left(x\frac{\mathrm{d}}{\mathrm{d}x} \right)^4 (R-1+x)^k \\
&= R^{-k} \lim_{x\to 1} \Big[kx(R-1+x)^{k-1} + 7k(k-1)x^2(R-1+x)^{k-2} \\
&\quad + 6k(k-1)(k-2)x^3(R-1+x)^{k-3} \\
&\quad + k(k-1)(k-2)(k-3)x^4(R-1+x)^{k-4} \Big] \\
&= kR^{-1} + 7k(k-1)R^{-2} + 6k(k-1)(k-2)R^{-3} \\
&\quad + k(k-1)(k-2)(k-3)R^{-4}.
\end{aligned} \tag{E.4}$$

$$\begin{aligned}
G_{2,2}^{k,R} &= R^{-k} \lim_{x,y\to 1} \left(x\frac{\mathrm{d}}{\mathrm{d}x} \right)^2 \left(y\frac{\mathrm{d}}{\mathrm{d}y} \right)^2 (R-2+x+y)^k \\
&= R^{-k} \lim_{x\to 1} \left(x\frac{\mathrm{d}}{\mathrm{d}x} \right)^2 \Big[k(R-1+x)^{k-1} + k(k-1)(R-1+x)^{k-2} \Big] \\
&= k(k-1)R^{-2} + 2k(k-1)(k-2)R^{-3} + k(k-1)(k-2)(k-3)R^{-4}.
\end{aligned} \tag{E.5}$$

APPENDIX F: Details of numerical simulation experiments

In order to reduce distraction in the main text, the details of the numerical simulation experiments have been omitted from the figure captions but are given below for the purpose of reproducibility. Due to the usual limitations imposed by available computing resources (memory and CPU) and programming skill, the system sizes were mostly limited to $pN = \alpha N^2 \leq 2^{28} \approx 2.7 \times 10^8$. Except for Fig. 1.4, all data refer to MG versions with $S = 2$ strategies per agent.

Figures in Chapters 1,2:

Simulations are of the on-line type. Experiments were carried out as standard with $N = 4{,}097$, and measurements were taken during an observation stage of 2,000 N individual iterations, following an equilibration stage of 1,000 N individual iterations. The only exception is Fig. 1.4, with system sizes $N \in \{513, 1{,}025, 2{,}049, 4{,}097\}$, and $S \in \{2, 3, 4\}$.

Figures in Chapter 3:

Simulations are of the on-line type. Experiments were carried out as standard with $N = 4{,}097$, and measurements were taken during an observation stage of $2{,}000N/\sqrt{\alpha}$ individual iterations, following an equilibration stage of $1{,}000N/\sqrt{\alpha}$ individual iterations. All data points shown are averages over five independent realizations of the disorder.

Figures in Chapter 4:

The numerical experiments concerned are of the batch type. Simulations were carried out as standard with $N = 4{,}097$, and measurements were taken during an observation stage of $1{,}000/\sqrt{\alpha}$ batch iterations (each of which being itself an average over $p = \alpha N$ different choices for the 'fake' market history), following an equilibration stage of $500/\sqrt{\alpha}$ batch iterations.

Figures in Chapter 6:
All simulations were carried out as standard with $N = 4{,}097$, unless indicated otherwise. Measurements in batch dynamics simulations were taken during an observation stage of $1{,}000/\sqrt{\alpha}$ batch iterations (each of which being itself an average over $p = \alpha N$ different choices for the 'fake' market history), following an equilibration stage of $500/\sqrt{\alpha}$ batch iterations. Measurements in on-line dynamics simulations were taken during an observation stage of $2{,}000N/\sqrt{\alpha}$ individual iterations, following an equilibration stage of $1{,}000N/\sqrt{\alpha}$ individual iterations. An exception is Fig. 6.4, where equilibration times were chosen as indicated in the caption.

Figures in Chapter 7:
Batch dynamics simulations were carried out as standard with $N = 4{,}097$, and measurements were taken during an observation stage of $1{,}000/\sqrt{\alpha}$ batch iterations (each of which being itself an average over $p = \alpha N$ different choices for the 'fake' market history), following an equilibration stage of $500/\sqrt{\alpha}$ batch iterations. An exception is Fig. 7.6, involving observations during $10{,}000$ batch iterations, starting from $t = 0$.

In the spherical model, simulations were carried out with $N = 500$, and measurements were taken during an observation stage of 250 batch iterations (each of which being itself an average over $p = \alpha N$ different choices for the 'fake' market history), following an equilibration stage of 250 batch iterations. Here all data are in addition averaged over 20 independent realizations of the disorder.

Figures in Chapter 8:
Here simulations are of the on-line type, with real histories. Experiments were carried out as standard with $N = 8{,}193$, and measurements were taken during an observation stage of $2{,}000N$ individual iterations, following an equilibration stage of $1{,}000N$ individual iterations. An exception is Fig. 8.4, where system sizes were chosen such that $2^M N = \alpha N^2 = 2^{28}$.

Figures in Chapter 9:
Batch dynamics simulations were carried out as standard with $N = 4{,}097$, and measurements were taken during an observation stage of $1{,}000/\sqrt{\alpha}$ batch iterations (each of which being itself an average over $p = \alpha N$ different choices for the 'fake' market history), following an equilibration stage of $500/\sqrt{\alpha}$ batch iterations. The on-line simulations of Figs 9.8 and 9.17 were carried out with system sizes as indicated. Here measurements were taken during an observation stage of $2{,}000N$ individual iterations, without a preceding equilibration stage.

References

[1] Challet, D., Marsili, M., and Zhang, Y. C. (2004) *Minority Games – Interacting Agents in Financial Markets*. Oxford: University Press.

[2] Feller, W. (1971) *An Introduction to Probability Theory and its Applications*. New York: Wiley.

[3] Van Kampen, N. G. (1981) *Stochastic Processes in Physics and Chemistry*. Amsterdam: North-Holland.

[4] Gardiner, C. W. (1994) *Handbook of Stochastic Methods*. Berlin: Springer.

[5] Abramowitz, M. and Stegun, I. A. (Eds.) (1970) *Handbook of Mathematics Functions*. New York: Dover.

[6] Gradshteyn, I. S. and Ryzhik, I. M. (1979) *Table of Integrals, Series and Products*. London: Academic Press.

[7] Menzel, D. H. (1960) *Fundamental Formulas of Physics*. New York: Dover.

[8] Arfken, G. (1985) *Mathematical Methods for Physicists*. London: Academic Press.

[9] Hinch, E. J. (1991) *Perturbation Methods*. Cambridge: University Press.

[10] Yeomans, J. M. (1992) *Statistical Mechanics of Phase Transitions*. Oxford: University Press.

[11] Plischke, M. and Bergersen, B. (1994) *Equilibrium Statistical Mechanics*. Singapore: World Scientific.

[12] Zinn-Justin, J. (1993) *Quantum Field Theory and Critical Phenomena*. Oxford: University Press.

[13] Bedeaux, D., Lakatos-Lindenberg, K., and Shuler, K. (1971) *J. Math. Phys.* **12** 2116–2123.

[14] Mantegna, R. N. and Stanley, H. E. (2000) *An Introduction to Econophysics*. Cambridge: University Press.

[15] Bouchaud, J. P. and Potters, M. (2003) *Theory of Financial Risk and Derivative Pricing — From Statistical Physics to Risk Management (2nd ed.)*. Cambridge: University Press.

[16] Arthur, W. (1994) *Am. Econ. Assoc. Papers Proc.* **84**, 406–411.

[17] Challet, D. and Zhang, Y.-C. (1997) *Physica A* **246** 407–418.
[18] Savit, R., Manuca, R., and Riolo, R. (1999) *Phys. Rev. Lett.* **82** 2203–2206.
[19] De Cara, M. A. R., Pla, O., and Guinea, F. (1999) *Eur. Phys. J. B*. **10** 187–191.
[20] Challet, D. and Marsili, M. (1999) *Phys. Rev. E*. **60** 6271–6274.
[21] Cavagna, A. (1999) *Phys. Rev. E*. **59** 3783–3786.
[22] Mézard, M., Parisi, G., and Virasoro, M. A. (1987) *Spin Glass Theory and Beyond*. Singapore: World Scientific.
[23] Fisher, K. H. and Hertz, J. A. (1991) *Spin Glasses*. Cambridge: U.P.
[24] Nishimori, H. (2001) *Statistical Physics of Spin Glasses and Informatio Processing*. Oxford: University Press.
[25] Challet, D., Marsili, M., and Zecchina, R. (2000) *Phys. Rev. Lett.* **84** 1824–1827.
[26] Marsili, M., Challet, D., and Zecchina, R. (2000) *Physica A* **280** 522–553.
[27] Marsili, M., and Challet, D. (2001) *Phys. Rev. E*. **64** 056138.
[28] Marsili, M., Mulet, R., Ricci-Tersenghi, F., and Zecchina, R. (2001) *Phys. Rev. Lett.*. **87** 208701.
[29] De Martino, A. and Marsili, M. (2001) *J. Phys. A* **34** 2525–2537.
[30] Heimel, J. A. F. and De Martino, A. (2001) *J. Phys. A* **34** L539–L545.
[31] De Dominicis, C. (1978) *Phys. Rev. B*. **18** 4913–4919.
[32] Heimel, J. A. F. and Coolen, A. C. C. (2001) *Phys. Rev. E*. **63** 056121.
[33] Coolen, A. C. C. and Heimel, J. A. F. (2001) *J. Phys. A: Math. Gen.* **34** 10783–10804.
[34] Challet, D., Marsili, M., and Zhang, Y.-C. (2000) *Physica A* **276** 284–315
[35] Coolen, A. C. C., Heimel, J. A. F., and Sherrington, D. (2001) *Phys. Rev. E*. **65** 16126.
[36] Sherrington, D., Coolen, A. C. C., and Heimel, J. A. F. (2002) *Physica A* **314** 83–91.
[37] Sherrington, D. and Galla, T. (2003) *Physica A* **324** 25–29.
[38] Galla, T., Coolen, A. C. C., and Sherrington, D. (2003) *J. Phys. A Math. Gen.* **36** 11159–11172.
[39] Kozlowski, P. and Marsili, M. (2003) *J. Phys. A Math. Gen.* **36** 11725-11737.
[40] De Martino, A., Giardina, I., and Mosetti, G. (2003) *J. Phys. A* **36** 8935-8954.
[41] Marsili, M. and Piai, M. (2002) *Physica A* **310** 234-244.
[42] De Martino, A. (2003) *Eur. Phys. J. B*. **35** 143–152.
[43] Johnson, N. F., Hui, P. M., Zeng, D., and Tai, C. W. (1999) *Physica A* **269** 493–502.
[44] Johnson, N. F., Hui, P. M., Zheng, D., and Hart, M. (1999) *J. Phys. A* **32** L427–L431.
[45] Challet, D. and Marsili, M. (2000) *Phys. Rev. E*. **62** 1862–1868.
[46] Lee, C.-Y. (2001) *Phys. Rev. E*. **64** 015102(R).

[47] Coolen, A. C. C. (2004) submitted to *J. Phys. A* `arXiv: cond-mat/0410335`.

[48] Johnson, N. F., Hart, M. L., Hui, P. M., and Zheng, D. (1999) *Theor. Appl. Finance*. **3** 443–450.

[49] Challet, D., Chessa, A., Marsili, M., and Zhang, Y.-C. (2001) *Quant. Fin.* **1** 168–176.

[50] Slanina, F. and Zhang, Y.-C. (1999) *Physica A* **272** 257–268.

[51] Challet, D., Marsili, M., and Zhang, Y.-C. (2001) *Physica A* **294** 514–524.

[52] Challet, D., Marsili, M., and Zhang, Y.-C. (2001) *Physica A* **299** 228–233.

[53] Challet, D. and Marsili, M. (2003) *Phys. Rev. E*. **68** 036132.

[54] Cavagna, A., Garrahan, J. P., Giardina, I., and Sherrington, D. (1999) *Phys. Rev. Lett.* **83** 4429–4432.

[55] Garrahan, J. P., Moro, E., and Sherrington, D. (2000) *Phys. Rev E*. **62** R9–R12.

[56] Sherrington, D., Moro, E., and Garrahan, J. P. (2002) *Physica A* **311** 527–535.

[57] Challet, D. and Zhang, Y.-C. (1998) *Physica A* **256** 514–532.

[58] Li, Y., Riolo, R., and Savit, R. (2000) *Physica A* **276** 234–264.

[59] Lo, T. S., Hui, P. M., Johnson, N. F. (2000) *Phys. Rev. E*. **62** 4393–4396.

[60] Hod, S. and Nakar, E. (2002) *Phys. Rev. Lett.* **88** 238702.

[61] Johnson, N. F., Hart, M., and Hui, P. M. (1999) *Physica A* **269** 1–8.

[62] Hart, M., Jefferies, P., Johnson, N. F., and Hui, P. M. (2000) *Phys. Rev. E*. **63** 017102.

[63] Hart, M., Jefferies, P., Hui, P. M., and Johnson, N. F. (2001) *Eur. Phys. J. B*. **20** 547–550.

[64] Johnson, N. F., Choe, S. C., Gourley, S., Jarret, T., and Hui, P. M. (2004) *Adv. Solid State Phys.* **44** 427–438.

[65] Galla, T. (2004) preprint `arXiv:cond-mat/0406158`.

[66] Hart, M. L., Jefferies, P., and Johnson, N. F. 2002 *Physica A* **311** 275–290.

[67] Challet, D., Marsili, M., and De Martino, A. (2004) *Physica A* **338** 143–150.

[68] Challet, D., De Martino, A., Marsili, M., and Perez Castillo, I. (2004 preprint) `arXiv:cond-mat/0407595`.

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